Chapter I

MODULATION, TRANSMISSION, and DEMODULATION

Jens Vidkjær
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I Modulation, Transmission, and Demodulation

Modulation is the process of transforming a baseband message to a form suitable for transmission through the channel in consideration. Demodulation is the reverse process of again recovering the original message. The scope of the demodulation depends on the type of data being send. In a radio telephony channel it may suffice at the receiver site to get an output with a power spectrum that contains the dominant part of the input power spectrum, in a television video channel it is important to reconstruct in time-domain the shape of the signal being send. In digital transmissions, the goal is to rebuild a logical bitstream represen-

![Power spectra for signals in the modulation and demodulation processes.](image)

Fig. 1 Power spectra for signals in the modulation and demodulation processes. It depends on the actual modulation type whether or not the spectra become similar with respect to shapes, symmetries, and bandwidths.

tation equivalent to the input stream. These and more distinct requirements are the background for the variety of modulating methods and modulator/demodulator circuits in use. However, all the types we shall consider below for RF communications have in common, that the modulating process transforms the low frequency baseband signal to a bandpass signal around a carrier frequency as sketched in Fig.1. The bandpass signal is the one actually transmitted to the receiver where the demodulator reconstruct the low-frequency baseband message. Obvious reasons for the procedure are that more baseband signals may be transmitted simultaneously through the same channel at different carrier frequencies. In radio-communication, moreover, efficient radiation and reception of signals through antennas require that the wavelength is comparable to their physical dimensions, so a move towards high frequencies makes the equipment manageable in size.

The scope of the presentation in this chapter is to provide a background for designing circuits and sub-systems that operate in the RF-frequency range. As will be apparent, there are still holes to be filled even to accomplish this limited goal. To get coverage of omissions and especially all the further aspects that are important to understand, plan, and design complete RF-communication systems, the reader should consult the comprehensive literature about the system aspects of communication. A few examples are given in the reference list where ref’s [1] and [2] encompass all types of modulations, [3] and [4] concentrate on digital communication, and [5] includes a detailed account on system design aspects.

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I-1 Basic Modulation Types and Concepts

The types of modulations suited for RF communication we consider are called continuous wave modulations, CW, where the baseband information is impressed upon a sinusoidal carrier wave of amplitude $A_{c0}$ and angular frequency $\omega_c$.

\[\text{carrier} : \quad y(t) = A_{c0} \cos \omega_c t, \quad (a)\]

\[\text{modulated carrier} : \quad y(t) = A(t) \cos (\omega_c t + \varphi(t) + \varphi_0). \quad (b)\]

The time dependencies of $A(t)$ and $\varphi(t)$ in the last equation contain the baseband message and the angle $\varphi_0$ represents an offset phase for the carrier compared to the timing of the baseband message. If there is no synchronism between the two, the offset may be set to zero without loss of generality. Eq.(1)(b) is called the envelope-phase representation of a modulated signal.

One major distinction between different modulation types is on how a baseband message contained in the signal $x(t)$ is impressed upon the modulated output $y(t)$. Here the basic extremes are amplitude modulation, where the phase does not depart from the carrier phase, and angle modulation, where the amplitude is kept constant.

![Fig.2 Examples of modulation waveshapes from a sinusoidal baseband signal $x(t)$. The modulation types shown are AM, amplitude, DSB(-SC) double sideband (suppressed carrier), PM, phase, and FM, frequency modulation.](image-url)
Amplitude Modulations, AM and DSB-SC

Examples of Amplitude Modulation, ϕ(t) = 0.

amplitude modulation, AM : \[ y(t) = A_c\phi (1 + m x(t)) \cos \omega_c t \]  \hspace{1cm} (a) \hspace{1cm} (2)

double-sideband, (suppressed carrier), DSB or DSB-SC : \[ y(t) = A_c\phi x(t) \cos \omega_c t \]  \hspace{1cm} (c) \hspace{1cm} (3)

Two common amplitude modulation methods are given above. Their waveshapes for a sinusoidal baseband signal are exemplified in Fig.2. The AM modulation in Eq.(2)(a) is intended to transfer the waveshape of the modulating baseband signal x(t) to the envelope of the carrier. Scaling of the signal levels is here quantified by the modulation index m. With a normalized baseband signal \(|x(t)| \leq 1\), the condition \(m \leq 1\) (or 100%) implies undistorted reproduction of the baseband signal to the carrier envelope. It is easy to reconstruct the baseband signal from an AM modulated wave in a receiver by the simple envelope detector circuit in Fig.3. Its detailed function will be considered later, but it should be realized that the low-pass filter bandwidth must exceed the envelope frequency. An AM modulated wave gets spectral components stemming from the baseband signal below and above the carrier. This is demonstrated for a sinusoidal baseband signal \(x(t) = A_x \cos \omega_x t\), \(A_x \leq 1\), and the trigonometric identity

\[ \cos a \cos b = \frac{1}{2} \cos (a + b) + \frac{1}{2} \cos (a - b) \]  \hspace{1cm} (3)

that yields

\[ y(t) \big|_{AM} = A_c\phi (1 + m A_x \cos \omega_x t) \cos \omega_c t \]

\[ = A_c\phi \cos \omega_c t + \frac{A_c\phi}{2} m A_x \cos (\omega_c - \omega_x) t + \frac{A_c\phi}{2} m A_x \cos (\omega_c + \omega_x) t. \]  \hspace{1cm} (4)
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Besides the distinct carrier term of frequency $\omega_c$ we get two so-called sidebands caused by the baseband signal, a lower sideband at the difference frequency $\omega_c - \omega_x$ and a corresponding upper sideband at $\omega_c + \omega_x$. A double-sided amplitude spectrum portray of this situation is shown in Fig. 4, where each term contributes a positive and a negative frequency component according to the Euler identities,

$$\cos a = \frac{1}{2} \left( e^{ja} + e^{-ja} \right), \quad \sin a = \frac{1}{j2} \left( e^{ja} - e^{-ja} \right).$$

With maximum undistorted modulation, i.e. $m=1$ and $A_x = 1$, the power of the AM modulated wave, say it is a voltage across a 1Ω resistor, becomes,

$$P_{AM} = 2 \frac{A_{c0}^2}{4} + 4 \frac{A_{c0}^2 m^2 A_x^2}{16} \bigg|_{m=1, A_x=1} = \frac{1}{2} A_{c0}^2 + \frac{1}{4} A_{c0}^2,$$

so at most 33% of the transmitted power contains the message from the baseband signal. However, the carrier power is required if simple envelope detection should be used in a receiver for the signal as suggested by the AM modulator and demodulator pair in the block scheme of Fig. 5 a,b. The ideal multiplying element is sometimes called a balanced modulator while its practical realization is called a balanced mixer. Note that the insertion of the carrier corresponds to adding a DC component to the baseband signal. To get explicit control of the carrier it is generally supposed that the baseband signal itself, $x(t)$, holds no DC term.

Compared to the result above, the DSB modulation from Eq.(2)(b) provides no distinct carrier component. This fact is often emphasized by adding the phrase "suppressed carrier" or SC. We have,
and the doublesided spectrum of this signal corresponds to the spectrum in Fig.4 with \( m=1 \) but without the two carrier components at angular frequencies \( \pm \omega_c \). We avoid assigning power to carrier components that bear no information, but the price paid is, that it becomes more complicated to get the baseband signal back in demodulation. As seen in the sinusoidal example of Fig.2, the envelope that would be sensed by an enveloped detector (shown in heavy line) is no longer the baseband signal. To detect the baseband from a DSB modulated signal it should again be multiplied by a carrier as shown in the DSB-SC modulator/demodulator pair in Fig.5 c,d. Although this at a first glance looks as simple as the AM pair, a prerequisite for proper operation is, that the two carrier oscillators are synchronized to run with equal phases. Let \( \theta(t) \) represents a possible difference in phase between the two - it will grow linearly with time if the frequencies differ - then the output from the demodulator multiplier becomes,

\[
u(t) = x(t) \cos \omega_c t \cos (\omega_c t + \theta(t)) = \frac{x(t)}{2} \cos \theta(t) + \frac{x(t)}{2} \cos (2 \omega_c t + \theta(t)) .
\]

The last resultant term holds high frequency components that are removed by low-pass filtering. The first term holds the recovered baseband signal, but if \( \theta(t) \) differs from zero the cosine factor either reduces or distorts it, so to get a predictable result the oscillator in the demodulator must be synchronized to the carrier of the received signal. A simple method is to let a fragment of the full carrier - a pilot carrier - follow the signal. This is done by adding

---

Fig.5 Block schemes for simple AM modulation (a), demodulation (b), and DSB-SC modulation (c) with the required synchronous demodulation (d).
Modulation, Transmission, and Demodulation

a constant $<1$ instead of 1 in Fig.5a. The receiver extracts the carrier for demodulation through a narrow bandpass filter as sketched in Fig.6.

![Demodulator principle for AM modulation with pilot carrier. It is extracted and amplified by a band-pass filter.](image)

Demodulation methods that require synchronization to the carrier are called coherent or synchronous. They are more fundamental than envelope detection. For instance, the AM signal may be coherently demodulated if the oscillators in Fig.5 a and d are synchronized.

### Angle Modulations, PM and FM

**Examples of Angle Modulation, $A(t) = A_{c0} - \text{constant}$**

\[
\begin{align*}
\text{phase modulation, PM} & : y(t) = A_{c0} \cos(\omega_c t + \beta x(t)) & (a) \\
\text{frequency modulation, FM} & : y(t) = A_{c0} \cos(\omega_c t + \varphi(t)), \quad \text{where} & (b) \\
\frac{d\varphi}{dt} & = \Delta \omega(t) = \Delta \omega_{\text{max}} x(t) = 2\pi \Delta f_{\text{max}} x(t). & (c)
\end{align*}
\]

Angle modulated signals hold no information in the amplitude and may take the form of phase modulation, PM, where the phase of the modulated signal deviates from the phase of the carrier in proportion to the baseband signal. With a normalized baseband signal, $|x(t)| \leq 1$, the scaling constant $\beta$ in Eq.(9)(a) is called the modulation index in angle modulations\(^1\). It specifies the maximum phase deviation from the carrier phase in either radians or degrees. The instantaneous frequency of the modulated signal is the time derivative of the total phase argument to the cosine factor in Eq.(1)(b). By frequency modulation, FM, the variation in instantaneous frequency from the carrier frequency is directly controlled by the baseband signal as shown by Eqs.(9)(b-c). The peak frequency deviation $\Delta f_{\text{max}}$ replaces here $\beta$ as the baseband signal scaling specification, but for sinusoidal modulation of a given frequency $\omega_x$, the two scale factors are related,

\(^1\) This is the common definition when modulating with analog signals. With digital signals the modulation index is often given as the maximum phase excursion over a bit period in units of $\pi$, cf. Eq.(36).
The generation of angle modulation may be based on a phase modulating principle as sketched in Fig.7. In FM the baseband signal has to be integrated before it is applied to the modulator. This method is called indirect FM in contrast to direct FM that is shown in Fig.8. Here the basic building block is a voltage controlled oscillator, VCO. It has an output signal of constant amplitude and an instantaneous frequency, which is controlled linearly around the center frequency \( \omega_c \) by an input voltage. The scaling factor \( K_V \) [Hz/Volt] is called the frequency gain in VCO terminology but it equals the peak frequency deviation \( \Delta f_{\text{max}} \) if the input \( x(t) \) is taken to be a voltage constrained to the interval \( \pm 1V \).

\[
x(t) = \cos \omega_c t \quad \rightarrow \quad \varphi = \Delta \omega_{\text{max}} \int \cos \omega_c t \, dt = \frac{\Delta \omega_{\text{max}}}{\omega_c} \sin \omega_c t ,
\]

\[
\beta = \frac{\Delta \omega_{\text{max}}}{\omega_c} = \frac{\Delta f_{\text{max}}}{f_c} .
\]

The generation of angle modulation may be based on a phase modulating principle as sketched in Fig.7. In FM the baseband signal has to be integrated before it is applied to the modulator. This method is called indirect FM in contrast to direct FM that is shown in Fig.8. Here the basic building block is a voltage controlled oscillator, VCO. It has an output signal of constant amplitude and an instantaneous frequency, which is controlled linearly around the center frequency \( \omega_c \) by an input voltage. The scaling factor \( K_V \) [Hz/Volt] is called the frequency gain in VCO terminology but it equals the peak frequency deviation \( \Delta f_{\text{max}} \) if the input \( x(t) \) is taken to be a voltage constrained to the interval \( \pm 1V \).

Phase modulation concepts are mostly used in conjunction with transmission of digital signals and we shall postpone exemplifications to the discussion of that topic. One method of demodulating\(^2\) FM signals in an asynchronous system is illustrated by Fig.9. The constant delay gives a phaseshift of \( \pi/2 \) (quadrature) at the carrier frequency. The difference term from the multiplication, cf.Eq.(3), transforms to a sine term through the identity

\[
\cos (a - \pi/2) = \sin a .
\]

---

\(^2\) In angle modulation, demodulation is often called phase or frequency discrimination.

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This component passes through the low-pass filter and approximates the original baseband messages according to Eq.(9)(c). The replacement of the sine term by its argument requires a small variation in phase over the period $\tau$, i.e. the baseband message held in $\phi(t)$ contains low-frequency components compared to the carrier frequency. The sum term from the multiplication gets twice the carrier frequency and it is removed by the low-pass filtering.

\[ \cos(\omega_c t + \phi(t)) \]

\[ \frac{1}{\tau} \cos(2\omega_c t + \ldots) + \frac{1}{2}\cos(\phi(t) - \phi(t - \tau) + \frac{1}{2}\pi) \]

\[ -\frac{1}{\tau} \sin(\phi(t) - \phi(t - \tau)) = -\frac{1}{\tau} [\phi(t) - \phi(t - \tau)] = -\frac{\tau}{\tau} \frac{d\phi(t)}{dt} \]

\[ = -\frac{1}{\tau} \Delta \omega_{\text{max}} x(t) \]

**Fig.9** Quadrature FM demodulator block scheme. With lumped element approximation of the delay, cf. example II-7-2 in chapter II, the principle is common in receivers for FM broadcast.

With sinusoidal baseband signals PM and FM waveshapes get similar appearances as seen in the two lower curves of Fig.2. Although they look completely different from amplitude modulated signals, angle modulated signals imply also lower and upper sidebands around the carrier in their spectra. Due to the nonlinear relationship through the cosine factors in Eqs.(9)(a-b), this is most easily demonstrated if $\beta$ is small, and we have the so-called narrowband PM or FM modulations. For $x(t) = A_x \cos \omega_c t$, $A_x \leq 1$, and using the identity,

\[ \cos(a + b) = \cos a \cos b - \sin a \sin b, \]

the FM modulated wave is written,

\[ y(t) |_{FM} = A_{\phi} \cos(\omega_e t) + \beta A_x \sin(\omega_c t) \]

\[ = A_{\phi} \cos(\omega_e t) \cos(\beta A_x \sin(\omega_c t)) - A_{\phi} \sin(\omega_e t) \sin(\beta A_x \sin(\omega_c t)) \]

By the estimations $\cos(a) \approx 1$ and $\sin(a) \approx a$ for $a \ll 1$ that are implied by the narrowband condition $\beta = \Delta \omega_{\text{max}} / \omega_c \ll 1$, and using the identity,

\[ \sin a \sin b = \frac{1}{2} \cos(a - b) - \frac{1}{2} \cos(a + b), \]

the FM waveform under narrowband conditions, NBFM, may be approximated,

\[ y(t) |_{NBFM} = A_{\phi} \cos(\omega_e t) - \frac{A_{\phi}}{2} \beta A_x \cos(\omega_e - \omega_c)t + \frac{A_{\phi}}{2} \beta A_x \cos(\omega_e + \omega_c)t. \]

Comparing narrowband FM with the AM modulated wave from Eq.(4) it is seen, that the modulation indexes $\beta$ and $m$ play the same role. Both types of modulations have an upper and lower sideband at frequencies $\omega_c$ apart from the carrier and the only principal difference is the sign of the lower sideband component, so the amplitude spectrum of the narrowband
sinusoidally modulated FM signal equals the AM spectrum of Fig.4 setting \( m = \beta \). It should also be realized that had we chosen baseband signal \( x(t) = A_x \sin \omega_x t \), the results above transfer directly to so-called narrowband PM. The only difference to FM is the interpretation of the modulation index \( \beta \) according to Eq.(11). Without the narrowband assumptions, PM and FM modulations show more spectral components than the two \( \omega_c - \omega_x \) and \( \omega_c + \omega_x \) terms above, even if the baseband still holds only a single tone. We shall see this in the following example.

**Example I-1-1 (Wideband FM and PM)**

A single tone baseband signal \( A_x \cos \omega_x t \) with scaling \( |A_x| \leq 1 \) is considered. We shall define an effective modulation index by

\[
\beta_{\text{eff}} = \beta A_x.
\]

If this index becomes so large that the assumption for Eq.(16) is no longer valid, expressions for the sinusoidally modulated waveshape must be based on series expansions for the baseband factors in the two terms of Eq.(14). We have, cf. ref.[6] Eqs.9.1.42 and 43,

\[
\begin{align*}
\cos(\beta_{\text{eff}} \sin \omega_x t) & = J_0(\beta_{\text{eff}}) + \sum_{n=2, \text{even}}^{\infty} 2J_n(\beta_{\text{eff}}) \cos n \omega_x t, \\
\sin(\beta_{\text{eff}} \sin \omega_x t) & = \sum_{n=1, \text{odd}}^{\infty} 2J_n(\beta_{\text{eff}}) \sin n \omega_x t.
\end{align*}
\]

![Fig.10 Bessel functions of the first kind and integer order. Tables may be found in ref.[6] but the functions are common in spreadsheets and mathematical programs like QuattroPro, Matlab, and Maple.](image-url)

*J.Vidkjaer*
Table I $J_n(\beta_{\text{eff}})$, significant expansion coefficients from Bessel functions.

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</tbody>
</table>

The expansion coefficients $J_n(\beta_{\text{eff}})$ are Bessel functions of the first kind and integer order. They depend on argument $\beta_{\text{eff}}$ as shown in Fig.10 and Table I. The outset from the narrowband case is easily observed in the figure since, for small $\beta_{\text{eff}}$, $J_0$ equals one, $J_1$ is proportional to $\beta_{\text{eff}}$ while all higher order functions approximate zero. With raising $\beta_{\text{eff}}$ terms of higher order than one get significance and more sidebands appear compared to the narrowband case in Eq.(16). Using the trigonometric identities for cosines and sine products, the FM signal for a cosine baseband tone - or the PM signal for a sine baseband tone - is obtained by inserting the expansions from Eq.(18) into Eq.(14) to yield

$$y(t)\big|_{\text{FM or PM}} = A_{c0}J_0(\beta_{\text{eff}})\cos(\omega_c t)$$

$$+ A_{c0} \sum_{n=2, \text{even}} J_n(\beta_{\text{eff}}) \left[ \cos(\omega_c t + n \omega_x t) + \cos(\omega_c t - n \omega_x t) \right]$$

$$+ A_{c0} \sum_{n=1, \text{odd}} J_n(\beta_{\text{eff}}) \left[ \cos(\omega_c t + n \omega_x t) - \cos(\omega_c t - n \omega_x t) \right]$$

$$- A_{c0} \sum_{n=-\infty} J_n(\beta_{\text{eff}}) \cos(\omega_c t + n \omega_x t).$$  \hspace{1cm} (19)$$

The last compaction includes negative order Bessel functions, where $J_{-n}(\beta)=(-1)^nJ_n(\beta)$, cf.[6] Eq.9.1.5. Since the sums run to infinity, the bandwidth of the signal is in principle unlimited. In practice, however, an effective bandwidth may be defined as the one containing frequency components up to an order where 99% of the theoretical total power is included. Since the power contents of a sinewave is independent of frequency and the FM or PM modulated signal has constant envelope, the total power is simply half the squared carrier amplitude $A_{c0}^2$. 

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I-1 Basic Modulation Types and Concepts

If $P_{ac,n}$ denotes the accumulated power of frequency components to order $n$ in size, the relative power accumulation is a function of the effective modulation index only,

$$p_{ac,n}(\beta_{eff}) = \frac{P_{ac,n}}{A_{\omega x}^2/2} = J_0^2(\beta_{eff}) + \sum_{i=1}^{n} 2 J_i^2(\beta_{eff}).$$  \hspace{1cm} (20)

Table II shows these relative powers for various integer modulation indexes. For each index, the first figure where power accumulation exceeds 99% is printed in bold. Calling the corresponding order $n_{99\%}$ we have the following rule

$$n_{99\%} = \beta_{eff} + 1 \hspace{1cm} (21)$$

The spacing between the components in the FM or PM signal is that of the baseband sinusoid, $\omega_x$, so the bandwidth required to transmit 99% of the theoretical power becomes

$$W_{99\%} = 2 \omega_x (\beta_{eff} + 1) = \begin{cases} 2 \left[ \Delta \omega_{\text{max}} A_x + \omega_x \right] & \text{FM modulation}, \\ 2 \omega_x \left[ \beta A_x + 1 \right] & \text{PM modulation}. \end{cases}$$  \hspace{1cm} (22)

The splitting on modulation types here is based on Eq.(11). In PM it is the maximum phase excursion $\beta$, corresponding to the maximum effective modulation index in the limit of $A_x=1$, which characterizes the modulation and is assumed constant. Therefore, the bandwidth in PM is proportional to the baseband frequency $\omega_x$. In the wideband case with $\beta>1$, the bandwidth also tends to become proportional with the amplitude $A_x$. In FM, the modulation is characterized by the maximum frequency deviation $\Delta \omega_{\text{max}}$ corresponding to a maximum amplitude of

Table II \hspace{0.2cm} $p_{ac,n}(\beta_{eff})$, accumulated relative power of frequency components to and including order $n$ in size. Figures in bold indicate the first 99% bound passing.
Fig. 11  dB scaled FM spectra for single tone baseband signal of frequency equal to the whole, one half, and one fourth of the maximum frequency deviation, $\Delta \omega_{\text{max}}$.

one. Keeping $\Delta \omega_{\text{max}}$ fixed, it is demonstrated by Fig. 11, that the bandwidth approaches twice this parameter when the baseband frequency becomes a smaller and smaller fraction of the maximum deviation. In the limit where $\Delta \omega_{\text{max}} \gg \omega_x$, so the effective modulation index is large, Eq(22) shows that the bandwidth also in FM becomes proportional to amplitude $A_x$.

So far, we have considered a single tone baseband signal. With composite signals matters become more involved. For a periodic baseband signal, which may be described by a Fourier series, the resultant modulated waveform is not a sum of terms of the type above, one set for each baseband component. The baseband signal enters phase terms in either sine or cosine functions, so the modulation process is nonlinear. Therefore, a correct calculation of the FM and PM signal spectra in that case should start with series expansions of the type in Eq.(18), which, in the particular case of a sinusoidal baseband signal, gave the Bessel function coefficient. Only few other cases provide tractable analytical solutions and we shall not discuss them here but refer to [1], sec. 5.3 or [2] sec.5.2. It may be argued, however, that with an arbitrary baseband signal $x(t)$, we may introduce modulation indexed according to

$$PM: \quad \beta_{\text{eff}} = \beta \max \{|x(t)|\}, \quad FM: \quad \beta_{\text{eff}} = \Delta \omega_{\text{max}} \max \{|x(t)|\}/W_x,$$  

(23)

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where $W_x$ is the bandwidth of $x(t)$. Using these definitions, Eq.(22) is known as Carson’s rule, and it is still useful for estimating the bandwidth of the modulated signal.

Example I-1-1 end

**Phasor Representation of Modulated Waveforms**

![Phasor representation showing how the lower and upper sideband components, $\ell$ and $u$, add to the carrier $A_{c0}$ in (a) AM modulation, and (b) narrowband FM. The modulated wave becomes $y(t) = \text{Re}(\xi)$.](image)

The contrast between the sideband terms in AM and narrowband FM is illuminated by the phasor representations in Fig.12. All terms in either Eq.(4) or Eq.(16) are taken as real valued projections of complex phasors at different angular frequencies. With AM the complex sum of the sideband components lie along the carrier. Due to the sign shift in the lower sideband term of narrowband FM, the sideband components here add to produce a component perpendicular to the carrier. The approximation in narrowband FM implies that the resultant length of the vector operations in Fig.12b stays close to the length of the carrier, i.e. $|\xi(t)| \approx A_{c0}$.

The phasor or complex number\(^3\) representation above is a quite general tool when dealing with more complicated modulation types. The common formalism begins with Eq.(1)(b), which by Eq.(13) may be rewritten,

$$y(t) = A(t) \cos(\varphi(t) + \varphi_0) \cos \omega_c t - A(t) \sin(\varphi(t) + \varphi_0) \sin \omega_c t$$

$$= x_i(t) \cos \omega_c t - x_q(t) \sin \omega_c t$$

where

$$x_i(t) = A(t) \cos(\varphi(t) + \varphi_0), \quad x_q(t) = A(t) \sin(\varphi(t) + \varphi_0).$$

\(^3\) The terms phasor, complex number, and vector representations of modulated signals are used synonymously in the literature.

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Eqs.(24) and (25) are called the quadrature-carrier or the I-Q representation of a modulated waveform. The naming refers to a coordinate system that rotates with the carrier, i.e. with the angular frequency $\omega_c$. The "in phase" $x_i(t)$ and "quadrature" $x_q(t)$ components are the instant projections of the modulated signal vector $\xi(t)$ on the axis along the carrier and the axis perpendicular (in quadrature) to the carrier. The two examples of Fig.12 have the components

\[
\begin{align*}
AM: \quad x_i(t) &= A_{c0} \left[ 1 + m x(t) \right], \\
&= A_{c0} \left[ 1 + m A_x \cos \omega_c t \right] \\
x_q(t) &= 0.
\end{align*}
\]

\[
\text{narrowband FM:} \quad x_i(t) = A_{c0}, \quad x_q(t) = A_{c0} A_x \sin \omega_c t.
\]

In a later context we shall benefit from a complex formalism for handling modulated signals, so it is worth already at this stage to observe that Eqs.(24) is equivalent to

\[
y(t) = \text{Re} \left\{ [x_i(t) + jx_q(t)] e^{j\omega_c t} \right\} = \text{Re} \left\{ \zeta(t) e^{j\omega_c t} \right\} = \text{Re} \left\{ \xi(t) \right\},
\]

where $\zeta(t) = x_i(t) + jx_q(t)$ is called the complex envelope of the modulated signal.

Fig.13 Quadrature modulator/demodulator pair. Note the sign shift in the demodulator sine carrier generation that corresponds to the subtraction in the modulator output.

Once the baseband signal is transformed into the components $x_i(t)$ and $x_q(t)$, the process of composing an arbitrarily modulated waveform and recover the two components again after undisturbed transmission may formally be made by the block scheme in Fig.13. The composite signals after the multipliers at the demodulator side include components based on the identity,

\[
\sin a \cos b = \frac{1}{2} \sin(a - b) + \frac{1}{2} \sin(a + b),
\]

\[
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\]
where the first sine term disappears if \( a = b \). Besides being a vehicle for analysis of CW modulated communication systems, practical circuits may be built directly on basis of the quadrature-carrier representation. They are called vector or I/Q modulators and an example is given below in Example I-1-2. Fig.14 shows a realization of the phase modulation from Eq.(9)(a) where preconditioning of the baseband signals is the mapping through nonlinear sine and cosine characteristics. A subset of this scheme is utilized in Fig.15 to make an indirect narrowband FM modulator of the type outlined by Fig.7.

**Fig.14** Phase modulator block scheme based on the quadrature-carrier resolution in Eqs.(24),(25). Note the nonlinear blocks holding cosine and sine functions.

**Fig.15** Indirect narrowband FM modulation scheme derived from Fig.14 by imposing the assumptions \( \cos(\phi) \approx 1, \sin(\phi) \approx \phi \) for \( \phi \ll 1 \).

In case of amplitude modulated signals, the general quadrature-carrier representation may seem overly complicated. With no phase excursions from the carrier, there is no need to map the original baseband signal \( x(t) \) onto in-phase and quadrature components. For instance, the DSB-SC scheme in Fig.5 could be either the transmission of \( x_i(t) \) or \( x_q(t) \) in Fig.13 setting the other one to zero. However, we could also use the quadrature principle to transmit two independent AM modulated signals simultaneously in the \( x_i(t) \) or \( x_q(t) \) branches, a technique that is called quadrature AM or QAM. If the originating baseband signals in this case have equivalent bandwidths, so do their modulated counterparts. Within the same frequency band of transmission two messages may now be send and demodulated.
Example I-1-2 (SSB, single sideband modulation)

Another way to obtain efficient use of a given frequency band is to transmit only one of the sidebands from a AM-DSB modulation, which is the so-called single-sideband modulation, SSB. To generate this signal, the modulator may again be constructed to manipulate the in-phase and quadrature components of the signal to the desired property. The principle is shown by Fig. 16 in the simple case of a sinusoidal baseband signal \( x(t) = A_x \cos\omega_xt \). Preprocessing is here an introduction of a 90° phaseshift between the baseband components of the in-phase and quadrature branches. The sign of the shift determines whether it becomes the upper or the lower sideband that is produced.

Fig.16 Phasing method of single sideband, SSB, modulation. The sign of the 90° phaseshifter determines whether the upper or lower sideband is produced.

Single sideband signals may be demodulated by the synchronous detector from Fig. 5d, if it is possible to reconstruct the carrier. With a reconstructed carrier of \( \cos(\omega_ct + \theta) \), \( \theta \) represents a possible synchronization error, the detected signal before low-pass filtering in case of upper sideband transmission becomes

\[
u(t) = A_x \cos(\omega_c t + \omega_x t) \cos(\omega_c t + \theta)
= \frac{A_x}{2} \cos \omega_x t \cos \theta + \frac{A_x}{2} \sin \omega_x t \sin \theta + \frac{A_x}{2} \cos(2\omega_c t + \ldots).
\]  

(29)

The first and the last terms here are equivalent to the two terms in the DSB-SC case from Eq.(8), where the last one is removed by filtering. The middle term is a new type of error - a distortion - that occur in SSB demodulation in addition to the baseband signal attenuation in the first term if \( \theta \) differs from zero due to inaccurate synchronization.

Synchronous demodulation of SSB signals has the drawback that any signal component in the frequency band corresponding to the removed sideband must be absent from the input. Suppose the input signal in addition to the desired upperband \( A_x \) component includes a foreign component \( A_{lo} \) in the frequency of the removed sideband. In that case the output from error free synchronous detection corresponding to Eq.(29) becomes

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Clearly, the desired output is now contaminated by the foreign component. It should be removed by filtering before the synchronous demodulator in order improve the situation. In the limit, however, the undesired component \( A_{lo} \) may represent unavoidable noise, so influence from the frequency range corresponding to the removed sideband cannot be totally neglected with this, in principle, rather simple demodulation scheme. To overcome that type of problems, i.e. to suppress responses in removed sideband, we must apply a demodulation scheme of the type that more closely follows the general demodulator pattern from Fig.13, for instance the SSB demodulator in Fig.17. As indicated by the figure, the quadrature demodulation allows for transmission of either upper or lower sidebands while the corresponding undesired frequency band, lower or upper sideband, cancels. It is the sign of the 90° phase shifter that determines the selection among the two possibilities.

SSB modulation is used extensively to transfer analog voice signals in radio telephony. It is less suited for digital data communications. To see this we consider a general input \( x(t) \) instead of the sinusoid that was used to illuminate the modulator operation in Fig.16. A 90° phase shifting that applies to all frequencies of \( x(t) \) is called the Hilbert transformation, sometimes denoted \( \hat{x}(t) \). It has transfer function \( H(\omega) = -j \) for \( \omega > 0 \) and \( H(\omega) = j \) for \( \omega < 0 \). The corresponding impulse response \( h(t) \) follows from the transform pair

\[
H(\omega) = -j \text{sgn}(\omega) \quad \text{h(t) = } \frac{1}{\pi} \frac{1}{t},
\]

where \( \text{sgn}(\omega) \) is the signum function. If the input signal is a pulse,
Modulation, Transmission, and Demodulation

The quadrature component of the baseband signal is expressed by the convolution

\[ x_q(t) = \hat{x}(t) \ast x(t) = \frac{1}{\pi} \int_{-T/2}^{T/2} \frac{1}{t - \tau} d\tau = \frac{1}{\pi} \log \left| \frac{T/2 - t}{T/2 + t} \right|. \]

With pulse input the quadrature component \( x_q(t) = \hat{x}(t) \) gets singularities at the pulse boundaries as shown in Fig.18. This means that the mixer in the quadrature branch of the modulator, the summing component and any subsequent power amplifiers momentarily should be capable of delivering infinitely large output signals, which clearly is an unrealistic requirement. Restricting output from the modulator distorts the resultant pulsed signals, and this is one of the reasons why SSB is not used in data communications. Another reason is that we get spectral efficiency corresponding to SSB with simpler means in the quaternary digital modulations that are presented in section I-3.

![Diagram](image.png)

**Fig.18** Baseband components for a pulse input to the SSB modulator in Fig.16. In-phase \( x_i(t) \) duplicates the pulse. Quadrature component \( x_q(t) \) approaches infinity at pulse boundaries.

---

**Example I-1-3 (vector modulator IC)**

Fig.19 shows an example integrated circuit implementation of a vector modulator, which includes an output amplifier after the summing point. RF integrated circuits have often differential or balanced signal paths, i.e. the signal are presented as the voltage between two dedicated conductors instead of the single ended voltage difference between one signal lead and a common ground. In integrated circuits signals are in most cases processed differentially, as it is difficult to establish a reference ground inside the circuits.

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*J.Vidkjær*
Silicon Bipolar Monolithic QPSK Modulator

Technical Data

Features
- I/Q Bandwidth: DC - 700 MHz
- LO Frequency Range: DC-2000 MHz
- Standby Mode - Pd 10 mW
- Low Cost Plastic Surface Mount Package

Applications
- Digital Cellular Radio (eg., GSM, ADC)
- RF Data Links
- Vector Generators
- AM Modulator
- Single Sideband Mixer

Description
The HPMX-2001 is a silicon monolithic quadrature phase shift keyed modulator in a plastic surface mount SO-16 package. It is designed for wide or narrow band applications and has a typical LO operating frequency range of DC-2000 MHz and typical I/Q bandwidth of DC-700 MHz.

The modulator may be operated in any combination of single or differential input/output configurations. The LO inputs are self biasing and have 50 Ohm termination impedances. The I & Q inputs are open base and require 2.5 V DC bias. This allows direct coupling to the I & Q inputs. The RF output is designed to drive a 50 Ohm load and has a saturated output power level of -5.5 dBm.

The internal signal paths of the modulator are fully differential to help reduce susceptibility to common mode noise. The small amplitude and phase imbalance of the device, 0.5 dB and 1 degree respectively, makes the modulator suitable for many communication applications. The HPMX-2001 also features a standby mode in which the device consumes only 10 mW of power.

The HPMX-2001 is manufactured using Hewlett-Packard’s 13 GHz P3, 25 GHz Fmax silicon bipolar integrated circuit process.

The HPMX-2001 is a plastic SO-16 package and Pin Configuration.

Fig.19  Vector modulator IC example, HPMX-2001 from Hewlett Packard, Communications Components, Designer’s Catalog 1993.
I-2 Binary Digital Modulations, ASK, PSK, FSK

The term analog modulation is used if the purpose of a modulation, transmission, and demodulation process is to reconstruct the baseband signal $x(t)$ in the receiver. Typical examples are $x(t)$ holding speech, music, or television signals in analog form. If the message to be impressed upon a modulated waveform originates from a digital signal, the goal at the receiver side is to minimize the probability of making wrong detections of the digits being send using initial knowledge about the waveforms to choose among.

If a digital message modulates a carrier bit by bit using two distinct waveforms, the process is called binary digital modulation. Examples are shown in Fig.20. They are the simple digital counterparts to the analog modulation types from Fig.2. The upper curve shows the baseband signal that imposes the bitstream to the modulators. This function is an important design objective, which is chosen here to let logical 1 translates to +1 and logical 0 to -1 throughout the duration of each bit. A baseband signal that this way stays constant in the bit period is termed NRZ, non return to zero.

Applying the digital baseband signal to the AM modulator from Fig.5a produces an amplitude shift keying signal, ASK. The particular example in Fig.20b has an AM modulation index $m=1$, so no signal is transmitted with logical zero. This is called on-off keying, OOK.
Phase modulation by a digital signal is called phase-shift keying, PSK. The case shown in Fig.20c corresponds to a phase difference of $\pi$, which for obvious reasons also is called phase-reversal keying, PRK, but it should be mentioned, that the term binary phase-shift keying, BPSK, often is associated with this particular choice of the phase difference between the two bit signals. Observe also that the same modulated wave results if the baseband modulates a carrier in DSB-SC amplitude modulation.

By frequency-shift keying, FSK, the instant frequency of the transmitted wave follows the baseband signal. A digitally controlled FSK wave may be produced by switching between two oscillators at fixed frequencies $\omega_1$, $\omega_2$. Applying the digital baseband to a FM modulator of the type in Fig.8 assures phase continuity like the example in Fig.20d. This is a common type of FSK in radio communication and it is abbreviated CPFSK, continuous-phase frequency-shift keying.

All examples in Fig.20 have the bit period $T_b$ set to an integral multiple of the carrier period. This is not a severe restriction as in most new communication systems all oscillator and timing signals are synthesized from the same source. We shall generally adopt the assumption below. The examples may be written in the form,

$$ y(t) = \sum_k s_b(t - kT_b), \quad s_b(t) = \begin{cases} s_1(t), & \text{logical 1 signal,} \\ s_0(t), & \text{logical 0 signal.} \end{cases} \quad (34) $$

where $k$ sums over all bits in the message. The terms in the sum are non-overlapping, because the two signal waveforms are zero outside a bit-period. The particular signals are,

**Binary Signal Waveforms**

$s_1(t), s_0(t)$ for $0 \leq t \leq T_b$

$s_1(t), s_0(t) = 0$ for $t < 0$ or $T_b < t$

**ASK**

$s_1(t) = A_{c0} \cos(\omega_c t - \frac{1}{2} \pi) \quad s_0(t) = 0,$

\[ -A_{c0} \sin \omega_c t \quad (a) \]

**PSK**

$s_1(t) = A_{c0} \cos(\omega_c t - \frac{1}{2} \pi) \quad s_0(t) = A_{c0} \cos(\omega_c t + \frac{1}{2} \pi)$

\[ = A_{c0} \sin \omega_c t, \quad = -A_{c0} \sin \omega_c t, \quad (b) \]

**FSK**

$s_1(t) = A_{c0} \cos(\omega_c t + \Delta \omega t + \theta_k), \quad s_0(t) = A_{c0} \cos(\omega_c t - \Delta \omega t + \theta_k)$,

\[ \omega_c = \frac{1}{2} (\omega_1 + \omega_2), \quad \Delta \omega = \frac{1}{2} (\omega_1 - \omega_2). \quad (c) \]

The phase offset of $\frac{1}{2} \pi$ in (a),(b) have been introduced to let the signals agree with the figures. They are drawn continuous at bit boundaries that encompass $t=0$. The FSK waveform in (c) includes a phase offset $\theta_k$ for the initial phase at bit number $k$. With CPFSK it serves the purpose of accumulating the phase changes throughout all foregoing bits.

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Constraining the phase offsets $\theta_k$ in FSK by a phase continuity requirement may be examined through the phase tree in Fig. 21. With initial phase deviation set to zero, the tree shows how the phase patterns may emerge when the instantaneous frequency - controlled by the baseband signal $x(t)$ - is integrated. Besides the frequency deviation $\Delta \omega = 2\pi \Delta f$, the scaling factor in digital FSK is often given through a modulation index $h$, which is defined by

$$ h = 2 \Delta f T_b, \quad (36) $$

A rational modulation index, i.e. $h = n/m$ with integers $n, m$, gives a finite number of modulo $2\pi$ phase offsets at bit boundaries. The simplest cases are indicated by Fig. 22. It flashes the state of the baseband signal onto an in-phase and quadrature baseband components plane at the bit boundaries, so it is an instant picture of the complex envelope vector from Eq. (27). We shall see below that to get conditions for optimal detection with digital modulations, the receiver must hold or reproduce the transmitted signals exactly in time. In that respect special cases with few initial states are attractable where Fig. 22b is known as Sunde’s FSK and Fig. 22d makes the foundation of the so-called minimum-shift keying, MSK.

Fig. 21 Phase tree in continuous phase frequency shift keying, CPFSK. The heavy line corresponds to the phase time function for the baseband signal in Fig. 20a.

Fig. 22 Examples of CPFSK phase-offsets at bit boundaries with $\phi_0 = 0$ for simple, rational modulation indexes $h$. 

*J. Vidkjær*
Optimal Detection of Binary Modulated Signals

A hypothetical scheme for demodulation binary digitally modulated signal is given in Fig.23. The incoming signal is applied to a filter with impulse response $h(t)$. The output of the filter is sampled after each bit period and the sampled value is compared to a threshold value $Z_{tr}$ to decide whether a $s_1(t)$ or a $s_0(t)$ signal was received. The sampling instants $t_k$ are assumed to be exactly synchronized to the bit periods in $y(t)$. This is not a straight-forward process, but its details are presently disregarded.

The response $z_b$ of the filter to input signal $s_b$ at the end of a bit period, where the filter is initially at rest, i.e. all capacitor voltages and inductor currents are zero, is given by the convolution integral,

$$z_b = s_b(t) \ast h(t) = \int_0^{T_b} s_b(\lambda) h(T_b - \lambda) \, d\lambda = \int_0^{T_b} s_b(T_b - \lambda) h(\lambda) \, d\lambda ,$$

(37)

---

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To make the detector insensitive to transmission disturbances or design approximations, the filter should be devised to maximize the distance between the two possible responses at the end of a period or, equivalently, the square of the difference

\[ |z_1 - z_0|^2 = \left| \int_0^{T_b} [s_1(T_b - \lambda) - s_2(T_b - \lambda)] h(\lambda) \, d\lambda \right|^2. \tag{38} \]

The squared integral complies with Schwartz inequality that states,

\[ \left| \int_{-\infty}^{\infty} V(t) W^*(t) \, dt \right|^2 \leq \int_{-\infty}^{\infty} |V(t)|^2 \, dt \int_{-\infty}^{\infty} |W(t)|^2 \, dt, \tag{39} \]

where \(V(t)\) and \(W(t)\) are two real or complex valued functions and where equality of the two sides is obtained if and only if \(W(t)\) is proportional to \(V(t)\). To use this result to the problem in Eq.(38) it is noticed, that the differences in integral bounds get no significance if \(h(t)\), like the functions \(s_1(t)\) and \(s_0(t)\), vanishes outside the bit interval. To maximize the left hand side of Eq.(38), the optimizing impulse response of the filter is a scaled, mirrored, and in principle complex conjugated version of the difference between the two basic signal waveforms. With this choice, the squared distance between the two responses may be expressed\(^4\),

\[ |z_1 - z_0|^2 \approx \left| \int_0^{T_b} (s_1(t) - s_0(t)) \, dt \right|^2 \int_0^{T_b} |h_{opt}(t)|^2 \, dt. \tag{40} \]

\[ h_{opt}(t) = K_0 \left\{ s_1^*(T_b - t) - s_0^*(T_b - t) \right\}. \tag{41} \]

\(K_0\) is an arbitrary gain factor for the filter that is implicitly contained also at the left hand side distances in Eq.(40).

Influence on the maximum detection distance from the characteristics of the original signal waveforms are more clearly revealed by rewriting the integral,

\[ \int_0^{T_b} |s_1(t) - s_0(t)|^2 \, dt = \int_0^{T_b} |s_1(t)|^2 \, dt + \int_0^{T_b} |s_0(t)|^2 \, dt - 2 \int_0^{T_b} \text{Re}\{s_1(t)s_0^*(t)\} \, dt \tag{42} \]

\[ = E_1 + E_0 - 2 \text{Re}\{\rho_{10}\} \sqrt{E_1E_0}. \]

Thinking of \(s_1\) and \(s_0\) being voltages across a 1Ω resistor, \(E_1\) and \(E_0\) are recognized as the energy per bit of the two signals. \(\rho_{10}\) is the correlation coefficient between the two signals, which is defined by,

---

\(^4\) Although the examples in Eq.(35) are all real, we maintain complex notation in basic developments to emphasize their scopes for future uses of the results.
In case the two waveforms have equal amplitudes, their energies are equal, and the detectable difference depends heavily on the correlation coefficient. If it is zero, the two signals are said to be orthogonal. If the correlation is -1, which implies the greatest detection distance, the signals are called antipodal.

\[ \rho_{10} = \frac{\int_{0}^{\tau_{b}} s_{1}(t) s_{0}^{*}(t) \, dt}{\sqrt{\int_{0}^{\tau_{b}} |s_{1}(t)|^2 \, dt \int_{0}^{\tau_{b}} |s_{0}(t)|^2 \, dt}} = \frac{\int_{0}^{\tau_{b}} s_{1}(t) s_{0}^{*}(t) \, dt}{\sqrt{E_{1} E_{0}}} \quad . \]  

(43)

In realistic environments the signal that reaches the detector is contaminated by noise, which, as shown by Fig.26, is modeled by a single input noise source \( n_y(t) \) that collects all noise contributions. It is a random process characterized by its spectral density \( S_{y}(\omega) \) and its amplitude probability distribution function. We shall go in more details with noise later, here it suffices to recall that if the input noise to a linear filter is Gaussian distributed so will be the output. The power of the output noise becomes the variance of \( n_z(t) \), \( \sigma_{z}^2 = <|n_z(t)|^2> \), but also the integral of the corresponding power distribution function across the entire frequency range. Assuming Gaussian input noise with constant double-sided spectral density \( \frac{1}{2}\eta \) [W/Hz], we have a so-called additive white Gaussian noise channel, AWGN. Now \( n_z(t) \) gets the power spectral density,

\[ S_{z}(\omega) = |H(\omega)|^2 S_{y}(\omega) = \frac{\eta}{2} |H(\omega)|^2 \quad . \]  

(44)

where \( H(\omega) \) is the filter transfer function. By this we get

\[ \sigma_{z}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{z}(\omega) \, d\omega = \frac{\eta}{2} \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 \, d\omega = \frac{\eta}{2} \int_{0}^{\tau_{b}} |h(t)|^2 \, dt \quad . \]  

(45)

The last expression follows from Rayleigh’s theorem,

\[ f(t) \leftrightarrow F(\omega) \quad \rightarrow \quad \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega \quad . \]  

(46)

\[ 5 ) \quad \eta \text{ stands for } kT, \text{ the product of Boltzmann’s constant } k=1.3807 \times 10^{-23} \text{[J/K]} \text{ and the absolute temperature } T \text{ in Kelvin [K]}. \]

---

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and the fact, that $H(\omega)$ and the impulse response $h_{\text{opt}}(t)$ make a Fourier transform pair. Inserting into Eq.(40) gives

\[
\left| \frac{z_1 - z_0}{2 \sigma_z} \right| \leq \sqrt{\frac{1}{2 \eta} \left[ E_1 + E_0 - 2 \text{Re} \left\{ \rho_{10} \right\} \sqrt{E_1 E_0} \right]}, \tag{47}
\]

where equality is cautioned by the use of $h(t) = h_{\text{opt}}(t)$ from Eq.(41) in calculating the noise variance by Eq.(45). We shall see below that this ratio is important for computing detection error probabilities. It gives the maximum detection distance relatively to twice the noise deviation at the output of an optimal filter expressed through the signal and noise conditions at the input to the filter.

Suppose the probabilities of transmitting a logical 1 and a logical 0 are $P(s_1)$ and $P(s_0) = 1 - P(s_1)$ respectively. Had there been no noise, the probability distribution of the sampled response $z_b(k)$ becomes the two impulses in Fig.27a. There is no possibility of making a wrong detection by the comparator of Fig.23, if the threshold is chosen anywhere in the interval $z_0 < Z_{tr} < z_1$. The presence of noise smears out the probability distribution of $z(k)$ as demonstrated by Fig.27b. It is given by

\[
p(z_k) = p_0(z_k) + p_1(z_k) = \frac{1}{\sigma_z} \mathcal{N} \left( \frac{z_k - z_0}{\sigma_z} \right) P(s_0) + \frac{1}{\sigma_z} \mathcal{N} \left( \frac{z_k - z_1}{\sigma_z} \right) P(s_1), \tag{48}
\]

where $\mathcal{N}(x)$ is the Gaussian, normal distribution function of zero mean and unit variance. This is the version that is commonly tabulated, and it relates to the Gaussian distribution with mean value $x_0$ and variance $\sigma_x^2$ through,

\[
p_G(x, x_0, \sigma_x) = \frac{1}{\sigma_x \sqrt{2 \pi}} \left| \begin{array}{c}
-\frac{(x-x_0)^2}{2 \sigma_x^2} \\
\left. \frac{e^{-x^2/2}}{\sqrt{2 \pi}} \right|_{x_0-0}^{\sigma_x-1}
\end{array} \right| = p_N(x) = \frac{1}{\sqrt{2 \pi}} e^{-\frac{x^2}{2}}. \tag{49}
\]

Fig.27  Probability distributions of the sampled signals excluding (a) and including (b) noise. The hatched areas in (b) represent the detection error probability, $P_{\text{error}}$ from Eq.(50).
The shape of the Gaussian distribution is summarized by Fig. 28 together with the two integrals $P(x_0)$ and $Q(x_0)$, which express the probabilities of the random variable $x$ being $< x_0$ or $> x_0$ respectively. The two cases are exclusive and exhaustive so their sum - the total area beneath the distribution curve - must be one. $P(x)$ and $Q(x)$ cannot be evaluated explicitly but may be looked up in standard mathematical tables, for instance Table 26.1 in ref. [6].

In the first term of Eq. (48), the Gaussian distribution is the density function of the conditional probability for $z(k)$ when a $s_0$ was sent. Likewise, the distribution in the second term is conditioned by a $s_1$ signal being sent. Since the Gaussian distributions are non-zero in the whole definition range of $-\infty < x < \infty$, the noise causes a non-zero probability of making a detection error for any comparator threshold setting $Z_{tr}$. This is illustrated by the hatched areas in Fig. 27b representing the probabilities of detecting logical 1 if the logical 0 signal $s_0$ was sent or detecting a logical 0 if the logical 1 signal $s_1$ was sent. Together, the two situations set the detection error probability that is expressed and evaluated though,

$$
P_{\text{error}} = P\{ z_b(k) > Z_{tr} \mid s_0 \text{ sent} \} + P\{ z_b(k) < Z_{tr} \mid s_1 \text{ sent} \}$$

$$= \int_{(Z_{tr}-z_0)/\sigma_z}^{\infty} p_N \left( \frac{z_b-z_0}{\sigma_z} \right) \frac{dz_b}{\sigma_z} P(s_0) + \int_{-\infty}^{(Z_{tr}-z_1)/\sigma_z} p_N \left( \frac{z_b-z_1}{\sigma_z} \right) \frac{dz_b}{\sigma_z} P(s_1)$$

$$= Q\left( \frac{Z_{tr}-z_0}{\sigma_z} \right) P(s_0) + Q\left( \frac{Z_{tr}-z_1}{\sigma_z} \right) P(s_1) \tag{50}$$

6 ) Another common way of expressing the probabilities uses the error function $\text{erf}(\cdot)$ and the complementary error function $\text{erfc}(\cdot)$. The two forms are related by

$$Q(x) \bigg|_{x<0} = \frac{1}{2} \text{erfc} \left( \frac{|x|}{\sqrt{2}} \right), \quad P(x) \bigg|_{x>0} = \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{|x|}{\sqrt{2}} \right) \right].$$

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where still \(P(s_0) = 1 - P(s_1)\). The Q functions hold the normalized integral defined in Fig. 28. As the probability of error depends upon the threshold setting \(Z_{tr}\), an optimal choice of this level should be the one that minimizes \(P_{\text{error}}\). Using the middle expression from Eq. (50), where each integral is differentiated with respect to one of its bounds, we get

\[
\frac{dP_{\text{error}}}{dZ_{tr}} = - \frac{1}{\sigma_z^2} P_N \left( \frac{Z_{tr} - z_0}{\sigma_z} \right) P(s_0) + \frac{1}{\sigma_z^2} P_N \left( \frac{Z_{tr} - z_1}{\sigma_z} \right) P(s_1) = 0. 
\]  

(51)

When the threshold lies between \(z_1\) and \(z_0\), as in Fig. 27, the slope of \(p_N\) in the first term is negative while it is positive in the second term. Under this condition 2nd order derivative of \(P_{\text{error}}\) is always positive and Eq. (51) corresponds to the desired minimum probability. Using the last part of Eq. (49) the result may be manipulated to yield the best threshold level

\[
Z_{tr} = \frac{1}{2} \left[ z_1 + z_0 \right] + \frac{\sigma_z^2}{z_1 - z_0} \log \frac{P(s_1)}{P(s_0)} \bigg|_{P(s_0) = 1 - P(s_1)}. 
\]  

(52)

Comparing error probabilities for different types of modulation on equal terms, it is custom to set the two bits in a digital transmission equally likely, i.e. \(P(s_1) = P(s_0) = \frac{1}{2}\). In that case the last term in Eq. (52) becomes zero and we get the plausible result that the threshold level should be centered between the two possible outcomes \(z_0\) and \(z_1\). Due to the symmetry of the Gaussian distribution the two error contributions from Eq. (50) may be collected to one term, so this important case is summarized by

\[
P(s_0) = P(s_1) = \frac{1}{2} : \quad Z_{tr} = \frac{1}{2} \left[ z_1 + z_0 \right] \quad \Rightarrow \quad P_{\text{error}} = Q \left( \frac{z_1 - z_0}{2 \sigma_z} \right) = Q \left( \sqrt{\frac{1}{2 \pi} \left[ E_1 + E_0 - 2 Re \{ \rho_{10} \sqrt{E_1 E_0} \} \right]} \right), 
\]  

(53)

where the last argument to the Q function is taken from the ratio in Eq. (47). Two points of importance for the use of the result should be stressed. First, there are made no assumptions about the particular waveshapes of the bit signals. Second, signals and noise are amplified equally so the filter gain factor \(K_0\) does no influence the error probability. However, the threshold in a practical realization of an optimal filter may depend on the gain. Eqs. (37) and (41) yield,

\[
z_1 = K_0 \int_{0}^{T_b} s_1(t) \left[ s_1^*(t) - s_0^*(t) \right] dt = K_0 \rho_{10} \sqrt{E_1 E_0}, 
\]  

(54)

\[
z_0 = K_0 \int_{0}^{T_b} s_0(t) \left[ s_1^*(t) - s_0^*(t) \right] dt = K_0 \rho_{10} \sqrt{E_1 E_0} - K_0 E_0. 
\]  

(55)

This result is again independent of the specific bit signal waveshapes of \(s_1\) and \(s_0\).
Errors in Optimal Detection of Binary Modulations

To compare threshold levels and error probabilities in optimal detection with different type of binary modulations, we need to know the two noiseless detection levels, \( z_1, z_0 \), the signal energies \( E_1, E_0 \), and their correlation coefficient \( \rho_{10} \). In a sinusoidal signal of constant amplitude \( A_{c0} \), voltage or current, the energy normalized to \( 1\Omega \) becomes,

\[
E_{c0} = \frac{T_b}{2} \int_0^{T_b} A_{c0}^2 \sin^2 \omega_c t \, dt = \frac{A_{c0}^2}{2} \int_0^{T_b} \left( 1 - \cos 2 \omega_c t \right) \, dt = \frac{A_{c0}^2}{2} T_b \left( 1 - \frac{\sin(2 \omega_c T_b)}{2 \omega_c T_b} \right)
\]

(56)

\[
- \frac{A_{c0}^2}{2} T_b \quad \text{for} \quad \omega_c T_b < \infty
\]

(57)

The mean power of a sine wave is usually given by \( \frac{1}{2} A_{c0}^2 \). This power is multiplied by the bit duration \( T_b \) to give the corresponding energy in the last expression. As seen, the conventional power formula requires, that the mean value is either calculated over long time or - due to the symmetry of the sinewave - over an integral number of quarter cycles. We have previously made the assumption that the bit period is an integral number of carrier cycles, so the last condition applies. In RF communication systems both conditions are commonly met, so Eq.(57) may be used to calculate signal energies in any case.

If logical 1’s and 0’s are equally likely, the mean energy per bit \( E_b \) in the signal reaching the detector becomes,

\[
E_b = \frac{1}{2} \left( E_1 + E_0 \right).
\]

(58)

The two first cases of modulated waves from Eq.(35) are now characterized,

**ASK - OOK modulation**

\[
E_1 = E_{c0}, \quad E_0 = 0, \quad E_b = \frac{1}{2} E_{c0}, \quad \rho_{10} = 0, \quad z_1 = 2K_0E_b, \quad z_0 = 0, \quad Z_{tr} = K_0 E_b,
\]

(59)

\[
\text{BER} = P_{\text{error}} = Q \left( \sqrt{\frac{E_{c0}}{2 \eta}} \right) = Q \left( \sqrt{\frac{E_b}{\eta}} \right),
\]

**PSK - PRK modulation**

\[
E_1 = E_{c0}, \quad E_0 = E_{c0}, \quad E_b = E_{c0}, \quad \rho_{10} = -1, \quad z_1 = 2K_0E_b, \quad z_0 = -2K_0E_b, \quad Z_{tr} = 0,
\]

(60)

\[
\text{BER} = P_{\text{error}} = Q \left( \sqrt{\frac{2E_{c0}}{\eta}} \right) = Q \left( \sqrt{\frac{2E_b}{\eta}} \right),
\]

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where BER stands for bit error rate. It is a common performance reference in comparisons including modulation methods that may extend to simultaneous transmission of more bits.

Noticing, for instance from Fig.28, that the Q-function decreases with growing argument, it is seen above, that the binary PSK-PKR modulations method has better error performance than the ASK-OOK scheme in both versions of $P_{\text{error}}$. The first one corresponds to equal maximum bit energies $E_{c0}$, i.e. equal amplitudes $A_{c0}$. This is relevant if it is the transmitter peak-power that imposes limitations. If they are set by regulations on the average power, where nothing is transmitted half the time in ASK-OOK, the $s_1$ signal amplitude is raised by a factor of $\sqrt{2}$ compared to the PSK modulated transmission, but even here the latter has an advantage. A final point of practical consequences that further favor PSK in this comparison is its zero threshold level that needs no adjustment, if the input energy level changes. With ASK-OOK adjustments in the $K_0$ factor through automatic gain control, AGC, must be devised.

With FSK there is one more design parameter to consider compared to the cases above, namely the frequency deviation $\Delta \omega$ that determines the separation between the two signals $s_1(t)$ and $s_0(t)$ and controls the correlation coefficient. It is assumed that the energies $E_1$ and $E_0$ are equal and expressible by Eq.(57), which also gives the mean energy $E_b = E_1 = E_0$. The correlation between the two signals becomes, cf. Eqs.(3), (35), and (43),

\[
\rho_{10} = \frac{1}{E_b} \int_0^{T_b} s_1(t) s_0^*(t) \, dt = \frac{A_{c0}^2}{E_b} \int_0^{T_b} \cos(\omega_c t + \Delta \omega t + \theta_k) \cos(\omega_c t - \Delta \omega t + \theta_k) \, dt
\]

\[
= \frac{1}{T_b} \int_0^{T_b} \cos(2\omega_c t + 2\theta_k) \, dt + \frac{1}{T_b} \int_0^{T_b} \cos(2\Delta \omega t) \, dt
\]

\[
= \frac{\sin(2\Delta \omega T_b)}{2\omega_c T_b} + \frac{\sin(2\Delta \omega T_b)}{2\Delta \omega T_b}.
\]

The first term in the last expression is exactly zero if the bit period is an integral multiple of the carrier period, otherwise it will lose significance with $\omega_c T_b \gg 1$. The same premises gave formerly Eq.(57). The last term may also disappear, but here the important cases are determined by the zeros of the sine function. Introducing the modulation index from Eq.(36), the correlation coefficient is expressed,

\[
\rho_{10} = \frac{\sin \left( \frac{2\pi h}{2\pi h} \right)}{2}, \quad \rho_{10} = 0 \quad \text{for} \quad h = \frac{P}{2}, \quad p = 1, 2, 3, \ldots.
\]

Using Eqs.(53) to (55) the properties of frequency-shift keying modulation from Eq.(35) (c) may be summarized,
Compared to Eq.(59) it is seen, that orthogonal FSK signals give the same error probability as the ASK-OOK modulation does. The effects of correlated FSK waveforms are demonstrated by Fig.29, which shows the bracket factor that makes the difference to ASK. The probability of error is clearly minimized with the most negative correlation, and this is obtained with non-orthogonal FSK signals having \( h = 0.715 \). The bracket factor is here 1.22 corresponding to a correlation of -0.22, but this is still far from the value of -1 that characterizes phase-shift keying PSK-PRK. The FSK optimum is often approximated by \( h = 3/4 \) with a corresponding correlation of -0.21. This is an approximation that is rational and could be implemented as a continuous-phase FSK signal with bit boundary offsets as shown in Fig.22(c). However, the considerations in this section concerning optimal detection cannot utilize any phase continuity in the detection. To do this requires memory beyond one bit period, which means a more complex detector. We shall consider an example below when minimum-shift keying, MSK, is discussed. MSK is a binary FSK modulation using \( h = 0.5 \), where the phase continuity exposes in the demodulation and provides a bit error rate equal to optimal PSK-PRK detection.

Fig.30 shows an overview of the error probabilities of the methods, which were considered in this section, i.e. Eqs.(59),(60), and (63). All the curves are similar but horizontally paralleled to each other as a consequence of the dB scaled arguments. The ratio \( E_b/\eta \) is sometimes denoted the signal-to-noise ratio, SNR, per bit.

\[
E_1 = E_{c_0}, \quad E_0 = E_{c_0}, \quad E_b = E_{c_0}, \quad \rho_{10} = \frac{\sin(2\pi h)}{2\pi h}, \quad z_1 = -z_0 = K_0 E_b (1 - \rho_{10}), \quad Z_{tr} = 0, \quad (63)
\]

\[
\text{BER} = P_{\text{error}} = Q \left( \frac{E_b}{\eta} \left[ 1 - \frac{\sin(2\pi h)}{2\pi h} \right] \right).
\]
Fig. 30  Error probabilities in the basic binary modulations. $E_b/\eta$ is the signal to noise ratio per bit, with mean bit energy $E_b$ and double-sided noise spectral density $\eta$. 

$P_{\text{error}}$
Optimal Detectors for Binary Modulations

Fig. 31 Realization of the optimal detector using two matched filters, one for each binary signal $s_1(t)$ and $s_0(t)$.

Realization of an optimal detector for binary modulation required a filter with impulse response equal to the difference between the two bit-signals $s_1, s_0$. This function may also be achieved by the scheme in Fig. 31, where there is a filter for each bit-signal, i.e.

$$h_{1,\text{opt}}(t) = K_0 s_1(T_b - t), \quad h_{0,\text{opt}}(t) = K_0 s_0(T_b - t). \quad (64)$$

Taking difference between the outputs from the two filters is equivalent to the basic optimum condition from Eq. (41). The structure is called the matched filter realization of the optimal detector, because the filters are defined by known waveforms the same way the matched filter concept is used elsewhere in signal processing. The outcomes of the two filters at the bit boundaries $t_k$ are expressible through the convolution

$$z_m(t_k) = h_{m,\text{opt}}(t) \ast s_n(t) = K_0 \int_{(k-1)T_b}^{kT_b} s_n(t) s_m(t) \, dt, \quad m,n \in \{1,0\}. \quad (65)$$

The detector structure in Fig. 32 redo the convolution integral directly. It evaluates like the numerator integral in the correlation coefficient from Eq. (43), so the structure in Fig. 32 is called the correlator realization of the optimal detector. The type of filtering applied here, with

Fig. 32 Correlator realization of an optimal detector for binary modulations. Local bit signal sources $s_1(t), s_0(t)$ must be synchronized to their incoming counterparts.

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All signals are real-valued in this section.

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integrators that are reset at bit boundaries immediately after their outputs are sensed by the sample and hold circuit, is called "integrate and dump" filtering.

The two realizations above perform identically if the basic optimum criteria in Eq.(64) are met and the necessary synchronization is perfect. In that respect the correlator version clearly emphasizes the requirement of signal waveform coherency in optimal detection as the detector contains synchronized replica of the two binary waveforms. In the matched filter version this property, especially the requirement to synchronization accuracy, is more implicitly contained in requirements to timing precision of the filter resetting and sampling times $t_k$. The problem may be illustrated by considering the reception of a $s_1(t)$ signal in either ASK or PSK. From Eqs.(35) and (64) we get

$$z(t) = s_1(t) \ast h_{1,\text{opt}}(t) = K_0 s_1(t) \ast s_1(T_b - t) = K_0 \int_{-\infty}^{\infty} s_1(\lambda) s_1(\lambda - t + T_b) d\lambda$$

$$= \begin{cases} 
0 , & t < 0 \text{ or } t > 2T_b, \\
K_0 A_{x_0}^2 \int_{0}^{t} \sin \omega_c \lambda \sin \left[ \omega_c (\lambda - t + T_b) \right] d\lambda , & 0 \leq t \leq T_b, \\
K_0 A_{x_0}^2 \int_{T_b}^{T_b - t} \sin \omega_c \lambda \sin \left[ \omega_c (\lambda - t + T_b) \right] d\lambda , & T_b \leq t \leq 2T_b.
\end{cases} \tag{66}$$

Fig.33 shows how the integral boundaries are taken from $\lambda$-axis ranges, where the functions in the convolution integral differs from zero. The integrals evaluate as, cf.[7] no.2.532,

$$\int \sin \omega_c \lambda \sin \left[ \omega_c (\lambda - t + T_b) \right] d\lambda = \frac{\lambda}{2} \cos[\omega_c (t - T_b)] - \frac{\sin[\omega_c (2\lambda - t + T_b)]}{4\omega_c} \tag{67}$$

In case of $\omega_c T_b > 1$ the first term becomes the dominant one. By this and the assumption of $T_b$ being an integral multiple of the carrier period, the non-vanishing part of the filter response to the $s_1(t)$ signal is approximated by,

Fig.33  Signal and impulse response for the convolution in Eq.(66). For $t < T_b$ the interval of non-zero product is $0 < \lambda < t$, for $t > T_b$ it is $t - T_b < \lambda < T_b$. 

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Fig. 34 Output from the $h_{1,\text{opt}}$ filter in Fig. 31. A $s_1(t)$ signal is received. $z_1$ should be detected, but is diminished $\Delta z$ by the timing error $\pm \Delta T_b$.

This result requires that the filter is initially at rest and not reset in the whole interval $0 < t < 2T_b$, where the convolution differs from zero. As seen in Fig. 34, the best sampling instant is the bit boundary $T_b$, where the response gets the expected maximum of $z_1 = \frac{1}{2}K_0 A^2 c_0 T_b^2 = 2K_0 E_b$, cf. Eqs. (59), (60). After sampling the filter is reset, so the remaining part of the response is not used and therefore shown dotted in the figure. Due to the oscillating nature of the output, the sensitivity of the sampling instant around the boundary follows a $\cos(\theta - \omega_c \Delta T_b)$ relationship. Timing errors do not influence the noise properties in the detector, so the net effect is a reduction of the difference $|z_1 - z_0|$ from Eq. (53) by a factor of $\cos \theta$ and, consequently, a greater error probability. Taking PSK-PRK as an example, Eq. (60) should be modified to read,

$$ P_{\text{error}} \bigg|_{\text{PSK-PRK}} = Q \left( \cos \theta \sqrt{\frac{2E_b}{\eta}} \right). $$

The problem of synchronizing and making resets is one of the aspects that favor the correlator method of building an optimal detector in RF circuits. As seen in Fig. 35, it is easy to make an integrator reset. Designing a filter with reset might be troublesome. Furthermore,

**Fig. 35** Integrator coupling (inverting) with an operational amplifier. Transistor T is normally open but it short-circuits in the reset period.
a RF filter must be causal to be physically realizable. The filter is defined by the signal waveshape, but most signals - in particular the simpler types we consider - do not have this property, so a hardware filter realization cannot be more than an approximation to optimal conditions from the very beginning.

Simplifications in the full optimal detector schemes are possible if the specific bit signal waveforms are constructed from a smaller number of more fundamental waves. In PSK-PRK or ASK-OOK modulation, only one correlator branch is required, because the two bit signals here are weighted versions of one basic signal, the sinusoidal carrier over a bit period. A received $s_1(t)$ signal of amplitude $A_{c0}$ causes the following outputs from the two branches of the detectors in Fig.32,

$$y(t) = s_1(t) = A_{c0} \sin \omega_c t$$

$$z_1(t_k) = K_0 A_{c0}^2 \int_0^{T_b} \sin^2 \omega_c t \, dt = \frac{K_0 A_{c0}^2}{2} T_b,$$

$$z_0(t_k) = K_0 A_{c0}^2 \int_0^{T_b} -\sin^2 \omega_c t \, dt = -\frac{K_0 A_{c0}^2}{2} T_b,$$

where it is assumed, that there are an integral number of carrier periods in $T_b$. The difference between the two outputs becomes positive, twice the magnitude of a single branch. With a $s_0(t) = -s_1(t)$ signal being received in PSK-ASK we get correspondingly,

$$y(t) = s_0(t) = -A_{c0} \sin \omega_c t$$

$$z_1(t_k) = -\frac{K_0 A_{c0}^2}{2} T_b,$$

$$z_0(t_k) = \frac{K_0 A_{c0}^2}{2} T_b,$$

and again the output difference becomes twice the size of a single branch, but now of negative sign. Had we used ASK-PRK the $s_0(t) = 0$ provides zero output here. The same results are also achieved using the detector in Fig.36, if we set the amplitude of the single local oscillator $A_{cl} = 2 A_{c0}$ to make comparisons on equal terms.

Fig.36 Detector for PSK-PRK or ASK modulation. The local oscillator must be synchronized to the carrier in $y(t)$ with $\theta = 0$, cf. Eq.(35).

The local oscillator in Fig.36 must be synchronized to the incoming carrier. To see the effect of a phase synchronization error $\theta$ in a correlator detector, consider the integrator output with a $s_1$ signal being received. It becomes
The integral evaluates like Eq.(67) and again, the first term becomes the dominant one. The detectable maximum value is reduced by the \( \cos \theta \) factor, so the error probability follows Eq(69) in this case too, although the premises are quite different as illuminated by Fig.37. The request for timing accuracy has moved to the phase of the local oscillator by \( \theta = \omega_c \Delta t \), whereas bit timing errors now are relatively insignificant.

Fig.37 Integrator output from the correlator detector, Eq.(72), using \( \omega T_b = 10\pi \) like Fig.34. The dotted curve shows the effect of a phase synchronization error \( \theta \).

Fig.38 Carrier recovery by squaring in PSK-PRK modulation. The output is synchronized to the incoming carrier with an ambiguity of \( \pi \).

Fig.39 Frequency divider principle. The limiter is a high-gain amplifier with symmetrically restricted output range. The toggle shifts state on positively going pulses.

\[
    z(t) = 2K_0A^2e_0 \int_0^t \sin \omega_c t \sin(\omega_c t - \theta) dt = K_0A^2e_0 \left[ t \cos \theta - \frac{\sin(2\omega_c t - \theta)}{4\omega_c} \right]. \quad (72)
\]
Synchronization in binary digital modulation causes often a phase ambiguity of $\pi$ in the locally generated carrier. The problem is illustrated by Fig.38 that shows a simple synchronization method for PSK-PRK. First the incoming signal is squared, which gives a DC component and a double carrier frequency component. The important point of the method is that the sign is independent of the signal being received, i.e. both $s_1(t)$ and the opposite $s_0(t)$ cause the same 2nd harmonic component, which follows the carrier phase. After removal of DC by filtering, the carrier is recovered through a frequency divider. In this operation there are no means to decide whether or not the result will be in phase or 180° out of phase with the original carrier. In plain terms considering the frequency divider method in Fig.39, the toggle that makes the division cannot distinguish between even or odd multiples $p$ of $2\pi$ phase offsets from the incoming signal. When $p$ is fixed initially, it should retain its value throughout the rest of the message. However, no synchronization is perfect. Input noise and noise from the circuitry contribute so-called phase or timing jitter and cause non-optimal performance. These effects are disregarded here, but we shall return to that question when more refined methods are discussed.

Although it might be possible to devise schemes that could exactly synchronize to the carrier, the alternative of making the transmission transparent to the 180° phase ambiguity by a method called differential encoding is often preferred. Before a digital message is modulated

![Diagram](image)

**Fig.40** Differential encoding (a) and decoding (b). Output from (a) toggles if $a_{\text{in}} = 1$ and stays constant on -1 (logical 0). The decoder output is independent of $b_{\text{in}}$’s polarity.

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8) More elaborate systems use phase-locked loop techniques for this purpose. They are considered later.

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on the carrier, it is transformed to another bitstream having the property, that logical ones are transmitted as a shifts in binary states from -1 to 1 or from 1 to -1, while logical zeros keep the binary state constant either 1 or -1. The reverse process of regenerating the original message after the detector will be independent of the polarity of the received bitstream or in turns, the 180° phase ambiguity of the recovered carrier. Fig.40 shows a differential encoder/decoder pair using exclusive or gates. The figure demonstrates, that the output \( b_{\text{out}} \) is indifferent to the polarity of the detected binary signal, \( b_{\text{in}} \) or \( b_{\text{in}}' \).

![Fig.41 Integrate and dump filter. The tracking stage follows the integrator output until reset. Then \( T_2 \) disconnects from the integrator and carries the integrator output voltage to the output S/H stage.](image)

It is important for proper operation of the integrate and dump filter, that the integrator is not reset before its output voltage is carried on to the sample and hold circuit. Fig.41 shows a possible realization where this is secured through an intermediate stage that tracks the integrator in most of the bit-period. Due to the inevitable delays of the inverters in the reset signal path, the tracking stage is uncoupled from the integrator just before the integrator resets. The final integrator output voltage is kept across \( C_{\text{aux}} \), while it is sensed by the output sample and hold stage.
Among the binary modulations considered so far, FSK has phase excursions from the carrier that needs projections on both the in-phase and the quadrature component of the carrier. To describe PSK-PRK or ASK, only one component was required. This leaves room for extensions with more efficient use of the frequency band around a given carrier.

**Quadrature Phase-Shift keying, QPSK**

The first step to improve spectral efficiency is to PSK-PRK modulate two bit sequences on the in-phase and quadrature carrier respectively. The two bitstreams may be taken from the same original data sequence as shown by the modulator in Fig.42, where even and odd numbered bits in NRZ form are directed to the in-phase and quadrature multipliers. The modulation is often called quaternary or quadrature phaseshift keying, QPSK, but due to the coincidence between PSK-PRK and AM-DSB modulations with binary data, it may also be considered as a quadrature amplitude modulation.

A detector for this type of modulation is shown in Fig.43. Each branch holds an optimal detector for the binary PSK modulated signals like the one in Fig.36. The integrators are reset in intervals of $T_s = 2T_b$ corresponding to the duration of bit signals in the baseband.

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in-phase and quadrature sequences $x_i(t), x_q(t)$. The intervals are twice the bit period $T_b$ of the input sequence $a_k$, and the final bits switch is operated in intervals of $T_b$ to reconstruct the original sequence. Provided that the synchronization is perfect, this detector is optimal in the same sense as before, i.e. for a given level of white Gaussian noise in addition to the input signal, it has the smallest probability of making a wrong detection. To see this, the deterministic part of the input signal that comes from the modulator may be written,

$$ y(t) = x_i(t) \cos \omega_c t - x_q(t) \sin \omega_c t $$

$$ = \sum_n s_{ib}(t - nT_s) - \sum_n s_{qb}(t - nT_s), \quad (73) $$

where $n$ sums over all pair of bits - or symbols - in the message. The bit signals for the in-phase and quadrature sequences are considered as projections on two basis signals $\varphi_i(t)$ and $\varphi_q(t)$ along the in-phase and quadrature carrier. They are defined by

$$ \varphi_i(t) = \begin{cases} \sqrt{\frac{2}{T_s}} \cos \omega_c t, & 0 \leq t \leq T_s, \\ 0, & \text{otherwise} \end{cases} \quad \varphi_q(t) = \begin{cases} \sqrt{\frac{2}{T_s}} \sin \omega_c t, & 0 \leq t \leq T_s, \\ 0, & \text{otherwise} \end{cases}, \quad (74) $$

The assumption of the carrier period being and integral multiple of the bit and in turns the symbol period makes the basis signals orthogonal. The scaling is chosen so they become orthonormal too, so they get the properties,

$$ \int_{-\infty}^{T_s} \varphi_i(t) \varphi_q(t) \, dt = \int_{0}^{T_b} \varphi_i(t) \varphi_q(t) \, dt = 2 \int_{0}^{T_s} \varphi_i(t) \varphi_q(t) \, dt = 0, \quad (75) $$

$$ \int_{-\infty}^{\infty} \varphi_i^2(t) \, dt = \int_{0}^{T_s} \varphi_i^2(t) \, dt = 1, \quad \int_{-\infty}^{\infty} \varphi_q^2(t) \, dt = \int_{0}^{T_s} \varphi_q^2(t) \, dt = 1. \quad (76) $$

Thereby, the bit signals may be written

$$ s_{ib}(t) = \pm \sqrt{E_b} \varphi_i(t), \quad s_{qb}(t) = \pm \sqrt{E_b} \varphi_q(t), \quad (77) $$

where positive and negative projections represent logical ones and zeros respectively. In units of the basis signals, the projections are the square roots of the bit energy $E_b$, which we have seen plays an important role for determining error probabilities when input noise is present. Working backwards, the instant amplitude $A_{cb}$ of either bit signal is expressed

$$ s_{ib} = \pm A_{cb} \cos \omega_c t = A_{cb} = \sqrt{\frac{2E_b}{T_s}} \quad \text{or} \quad E_b = \frac{A_{cb}^2}{2} T_s, \quad (78) $$

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which is similar to the relationship that was previously used by Eq.(57), taking into account that the energy now corresponds the period $T_s = 2T_b$.

It is the orthogonality condition in Eq.(75) that makes the detector in Fig.43 optimal. The input to the two branches are the combined $s_{ib}(t)$ and $s_{qb}(t)$ signals, but with perfect synchronization of the local oscillators the upper and lower branch suppress completely the $s_{qb}$ or $s_{ib}$ respectively. The developments in the forgoing section applied to the lower branch with $s_b = \pm A_{cb} \sin \omega_c t$, but all calculations could equally well have used $s_b = \pm A_{cb} \cos \omega_c t$. Therefore, both branches in Fig.43 are optimal and have equal bit error probabilities or error rates. Since the number of bits received by the complete QPSK demodulator is twice the number of bits in one of the branches, the total bit error rate of the modulator is equal to the rate of a single branch and we have, cf. Eq.(60),

$$BER_{QPSK} = BER_{PSK-PRK} = Q \left\{ \frac{2E_b}{\eta} \right\} , \quad (79)$$

While $E_b$ is the total energy of the binary PSK signal, the total energy of a QPSK signal - called the symbol energy $E_s$ - is twice of that. To acquire the potential double transmission rate of QPSK modulation within the bandwidth of a PSK system, the symbol time in QPSK must be the bit time in PSK. If the two systems should perform equally with respect to bit error rates, the output power $P_{sig}$ of the QPSK transmitter must be twice the power of the PSK-PRK system.

The power, energy and amplitude relationships in a QPSK signal are illuminated by Fig.44, which shows the two common graphical representations of modulated signals. Fig.44b displays the in-phase and quadrature components of the signal. It is a mapping of the complex envelope to the signal that was introduced earlier, cf.p.14. The plot in Fig.44a is called a signal space diagram and shows the possible signal states in basic orthonormal coordinates.

Fig.44 Signal space (a) and in-phase, quadrature component (b) representation of a QPSK modulated signal. The signal state may change instantly along the dashed lines at symbol boundaries.
**Offset Quadrature Phase-Shift keying, OQPSK**

Constant amplitude signals are preferred in many communication systems. They may ease construction of demodulators and allow employment of nonlinear, high efficiency amplifiers in the system. The QPSK signal has clearly this merit and would retain it through transmission from the modulator to the demodulator, had there been no filtering or other bandwidth limitation in the path between the two. The failure of this illusory assumption has the consequence that abrupt changes in the signal phase cause amplitude variations. When the message state change in a QPSK signal, the phase may instantly exhibit 90° and 180° jumps. The 180° jump is avoided if one of the signal component is offset or staggered by a bit period in a modulation format called offset quaternary phase-shift keying, OQPSK. It is produced by a the modulator in Fig.45.

![OQPSK modulator](image)

If there are no bandwidth restrictions, the properties of a OQPSK system are basically the same as the properties of a QPSK system. Due to the orthogonality that stems from the in-phase and quadrature carriers, the in-phase and quadrature branches could be considered separately before. This is also conceivable with OQPSK, because the orthogonality applies to a bit period so even with offset, the in-phase and quadrature signals do not interfere in demodulation. The only modification to the demodulator in Fig.43 for handling OQPSK signals is, that the two integrator resettings must be offset by a bit period. All other conditions are equal, in particular, the power spectra of the two signals are the same. At a first glance this may seem strange, as the maximum phase jump in the OQPSK signal has halved. However, the effect of the halving is balanced by the fact, that the jumps now occur with double rate because the modulated wave may change state in intervals of $T_b$ instead of $T_s=2T_b$ as seen in the OPQSK example of Fig.46.

9 ) The power spectra are presented on page 47.

10 ) Fig.46, and later Fig.47, are drawn with a carrier periods equal to one bit interval, i.e. $\omega_c=2\pi/T_b$. This is sufficient to meet all assumptions about orthogonality and it keeps the figures clear. In practical narrowband applications there must be many more carrier periods per bit interval.

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Fig. 46  Waveshapes in OQPSK signal generation by the modulator in Fig. 45.

\[ x_i(t) = A_{cb}b_{evn}(t) \]
\[ x_q(t) = A_{cb}b_{odd}(t) \]
\[ s_i(t) = x_i \cos \omega_c t \]
\[ s_q(t) = x_q \sin \omega_c t \]
\[ y(t) = s_i(t) - s_q(t) \]

Fig. 47  Shaped OQPSK modulation from the modulator in Fig. 48. Equivalency to MSK is illuminated by the continuous phase and the two frequencies \( \omega_1 \) and \( \omega_2 \) in the output.

\[ x_i(t) = A_{cs}b_{evn}(t) \sin \Omega t \]
\[ x_q(t) = A_{cs}b_{odd}(t) \cos \Omega t \]
\[ s_i(t) = x_i \cos \omega_c t \]
\[ s_q(t) = x_q \sin \omega_c t \]
\[ y(t) = s_i(t) - s_q(t) \]
Minimum-Shift Keying, MSK

Fig. 48  Modulator for shaped OQPSK modulation. The baseband signals from the modulator in Fig.45 are shaped to half-sines before they reach the multipliers.

Shaping of the OQPSK baseband in-phase and quadrature signals by a half period sinetip provides a very smooth output signal. The modulation may be generated by inserting pulse shaping functions in the baseband branches of the OQPSK modulator as shown in Fig.48. Corresponding waveshapes in the modulation process are illuminated by Fig.47. The main characteristics of the output signal is the constant amplitude and a continuous-phase. We shall see that this technique actually provides a binary CPFSK wave with modulation index \( h = 0.5 \) known as minimum-shift keying, MSK. The minimum term refers to the observation already made in Fig.29, namely \( h = 0.5 \) is the smallest FSK index where the corresponding frequency modulated signals are orthogonal.

Before expanding on MSK properties, a few observations based on shaped OQPSK considerations are in place. The first concerns the orthogonality of the signal components. One way of describing the modulated signal by in-phase and quadrature components is,

\[
y(t) = s_i(t) - s_q(t) = x_i(t) \cos \omega_c t - x_q(t) \sin \omega_c t,
\]

\[
x_i(t) = A_{cs} \cdot b_{\text{even}}(t) \mid \sin \Omega t \mid,
\]

\[
b_{\text{even}}(t) = \sum_{k_{\text{even}}} b_k p_{2T_b}(t - kT_b),
\]

\[
x_q(t) = A_{cs} \cdot b_{\text{odd}}(t) \mid \cos \Omega t \mid,
\]

\[
b_{\text{odd}}(t) = \sum_{k_{\text{odd}}} b_k p_{2T_b}(t - kT_b),
\]

\[
\text{where } \quad \Omega = \frac{\pi}{2T_b}, \quad b_k \in \{1, -1\}, \quad p_{2T_b}(t) = \begin{cases} 1, & 0 \leq t \leq 2T_b, \\ 0, & \text{otherwise}. \end{cases}
\]

11) Every author has his own approach to MSK modulation and this one makes no exception. A common, different outset is to let the shaping sinewaves originate from a running oscillator, i.e. without taking absolute values in Eqs.(81)(82). The present form has the advantage of being directly expandable from the OQPSK modulator, and to provide a basis for future refinements with other shaping functions.

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Since $|b_{evn}(t)|=|b_{odd}(t)|=1$, the integral of the in-phase and quadrature signals over a bit period becomes,

$$
\int_{kT_b}^{(k+1)T_b} s_i(t)s_q(t) \, dt = \pm A^2_{cs} \int_0^{T_b} \sin \Omega t \cos \omega_c t \cos \Omega t \sin \omega_c t \, dt
$$

$$
= \pm \frac{A^2_{cs}}{8} \int_0^{T_b} \left[ \cos(2\omega_c t - 2\Omega t) - \cos(2\omega_c t + 2\Omega t) \right] \, dt
$$

$$
= \pm \frac{A^2_{cs}}{8} \left[ \frac{\sin(2\omega_c t - 2\Omega t)}{2\omega_c - \Omega} - \frac{\sin(2\omega_c t + 2\Omega t)}{2\omega_c + \Omega} \right]^{T_b}_0.
$$

Each term in the result vanishes if - as presupposed - $\Omega = \pi/2T_b$ and $\omega_c = n\pi/T_b$, so orthogonality is maintained, and a detector of the type in Fig.49 will be optimal, if the synchronization is ideal. Besides the reset staggering that was also required in OQPSK, the $|\sin \Omega t|$ and $|\cos \Omega t|$ generators in the receiver are required to correlate with the half-sine baseband waveforms. As the basic error probability expressions, cf. page 28, depend solely upon the energy in the signal regardless of waveform, the bit error rate is unaltered from OQPSK, i.e.

$$
BER_{\text{QPSK}} = BER_{\text{OQPSK}} = BER_{\text{MSK}} - Q\left(\frac{\sqrt{2E_b}}{\eta}\right).
$$

The bit energy to be used in MSK is given by,

$$
E_b = \int_0^{2T_b} s_i^2(t) \, dt = A^2_{cs} \int_0^{2T_b} \cos^2 \omega_c t \sin^2 \Omega t \, dt = \frac{A^2_{cs}}{2} T_b - \int_0^{2T_b} s_q^2(t) \, dt
$$

The corresponding total signal power and amplitude $A_{c0}$ are

$$
P_{\text{sig}} = \frac{A^2_{c0}}{2} = \frac{2E_b}{2T_b} \quad \rightarrow \quad A_{c0} = \sqrt{\frac{2E_b}{T_b}} = A_{cs}.
$$

Fig.49 Detection principle for shaped OQPSK. Correlators for the half-sine shaping functions are shown separately in each branch.
To see the effect of smoothing the waveform through shaped OQPSK or MSK modulations compared to QPSK and OQPSK, we should consider their power spectral densities. A baseband signal,

$$s_b(t) = \sum_k b_k p_b(t - kT_s), \quad (88)$$

where \(b_k\) is a random binary NRZ sequence of rate \(1/T_s\) with equally probable 1’s and -1’s, and where \(p_b(t)\) is the bit signal shaping function, has the power spectral density

$$S_b(\omega) = \frac{1}{T_s} |P_b(\omega)|^2. \quad (89)$$

\(P_b(\omega)\) is the Fourier transform of \(p_b(t)\). \(S_b(\omega)\) is called the spectral density of the equivalent baseband signal. When this baseband signal modulates a sinusoidal carrier of frequency \(\omega_c\) - say the in-phase component - the modulating property of the Fourier transform implies, that the modulated output gets the power spectral density

$$S_{y,i}(\omega) = \frac{1}{4} \left[ S_b(\omega + \omega_c) + S_b(\omega - \omega_c) \right]. \quad (90)$$

Modulating two equivalent but statistically independent baseband sequences in quadrature doubles the resultant spectral density,

$$S_{y,iq}(\omega) = \frac{1}{2} \left[ S_b(\omega + \omega_c) + S_b(\omega - \omega_c) \right] \quad (91)$$

The pulse to be used for calculating the QPSK and OQPSK spectra is a rectangular pulse of width \(T_s = 2T_b\) and height \((E_b/T_b)^{1/2}\), cf. Eq.(78). It gives,

\[
\text{QPSK, OQPSK} \quad P_b(\omega) = \sqrt{\frac{E_b}{T_b}} 2T_b \frac{\sin(\omega T_b)}{\omega T_b} \quad \rightarrow \quad S_b(\omega) = 2E_b \left(\frac{\sin(\omega T_b)}{\omega T_b}\right)^2. \quad (92)
\]

With MSK, the baseband pulse is a half-sine of width \(2T_b\) and, according to Eq.(87), height \((2E_b/T_b)^{1/2}\). Moving to a position of even symmetry we get,

\[
\text{MSK} \quad P_b(\omega) = \sqrt{\frac{2E_b}{T_b}} 2 \int_0^{T_b} \cos\left(\frac{\pi t}{2 T_b}\right) \cos \omega t dt \quad (93)
\]

\[
= \sqrt{\frac{2E_b}{T_b}} \left[ \frac{\sin(\omega T_b - \frac{\pi}{2})}{\omega - \frac{\pi}{2 T_b}} + \frac{\sin(\omega T_b + \frac{\pi}{2})}{\omega + \frac{\pi}{2 T_b}} \right] = \sqrt{\frac{2E_b}{T_b}} \frac{\cos(\omega T_b) 4 T_b}{\pi \left[1 - \left(\frac{2 \omega T_b}{\pi}\right)^2\right]} \quad \rightarrow
\]
The two equivalent baseband spectra from Eqs.(92) and (94) are shown in Fig.50 using $E_b=1$. The MSK signal has a broader main lobe than the QPSK, OQPSK signal but also the great advantage, that outside this lobe, the spectrum rolls off much faster than QPSK and OQPSK. In practical communication systems bandwidth is a limited and regulated resource. If the spectrum of a modulated wave is restricted to a certain bandwidth by filtering after the modulator, the transmitted wave becomes distorted. Thereby we get so-called intersymbol interference in the demodulation which increases the bit error rate compared to the optimal conditions. It should be clear, however, that the more the natural power spectrum of the signal is concentrated around its center, the less will be the number of errors introduced by output filtering.

$$S_b(\omega) = \frac{16}{\pi^2} E_b \left( \frac{\cos(\omega T_b)}{1 - \left( \frac{2 \omega T_b}{\pi} \right)^2} \right)^2.$$  \hfill (94)

Fig.50 Power spectral densities for QPSK, OQPSK and MSK modulations. In MSK 99% of the signal power is contained within normalized frequency bounds of $\pm 1.17$. In QPSK, OQPSK the same figure is approximately $\pm 8$.

Above it was established that the modulator in Fig.48 transmits a data sequence that the demodulator in Fig.49 detects optimally. Now we shall show, that the resultant modulated wave is equivalent to minimum-shift keying, MSK. According to the introduction in section I-2, a binary CPFSK signal may be generated with a direct FM modulator like the one in Fig.51, where the binary digits are directly mapped on two frequencies $\omega_1, \omega_2$. The two frequencies are easily recognized in the modulated wave $y(t)$ in the example of Fig.47, but it
is also clear, that the data-sequence obtained taking $\omega_1$ as logical 1 and $\omega_2$ as logical 0 - or visa versa - is not the same as the input data-sequence to the shaped OQPSK modulator. To use the latter as a substitution for the direct FM modulator requires a remapping of the input bit sequence. It is the digital counterpart to the baseband integration, which is necessary, when a phase modulator is used to produce analog FM modulation, cf. page 7.

Expressing the output from the direct FM modulator by its in-phase and quadrature components provides,

$$y(t) = A_e \cos(\omega_c t + \varphi_0 + \varphi(t)) = A_e \cos(\omega_c t + \varphi_0) + \pi h\int x(t) dt,$$

where the phase function $\varphi(t)$ contains the integral of the frequency deviations, which are controlled by the input bit sequence. $\varphi_0$ is an arbitrary phase offset that must be fixed later, when the timing of the signal is compared to the similar shaped OQPSK signal. With $h=0.5$ the frequency deviation from the carrier given through Eqs.(35)c and Eq.(36) corresponds to the frequency of the half-sine shaping function above,

$$\Delta \omega = \frac{\pi h}{T_b} = \frac{\pi}{2T_b} = \Omega.$$

Assuming $\varphi(0)=0$, the phase function becomes,

$$\varphi(t) = \Omega \int_0^t a_{in}(t) dt, \quad a_{in}(t) = \sum_k a_k p_{T_b}(t - kT_b),$$

where $a_k$ represents the input bits in NRZ form, i.e. $a_k \in \{1,-1\}$, and $p_{T_b}(t)$ is a pulse of height 1 and length $T_b$. The phase function follows a pattern in the phase tree from Fig.21. In the $k$'th bit interval, the function may also be expressed,

$$\varphi(t) = \theta_k + a_k \Omega \tau, \quad \tau = t - kT_b, \quad \begin{cases} kT_b \leq t \leq (k+1)T_b, \\ 0 \leq \tau \leq T_b, \end{cases}$$

where $\theta_k$ is the phase offset at the $k$'th bit boundary $kT_b$. It is given and updated through,
\[ \theta_k = \frac{\pi}{2} \sum_{i}^{k-1} a_i, \quad \theta_{k-1} = \theta_k + a_k \frac{\pi}{2}. \]  

(99)

At even and odd bit boundaries the phase offset is constrained to the values, cf. Fig.21,

\[ \theta_{k \text{ even}} = \begin{cases} 0 + 2p\pi, & \rightarrow \cos \theta_k = 1, \\ \pi + 2p\pi, & \rightarrow \cos \theta_k = -1, \end{cases} \]

\[ \theta_{k \text{ odd}} = \begin{cases} \frac{\pi}{2} + 2p\pi, & \rightarrow \sin \theta_k = 1, \\ -\frac{\pi}{2} + 2p\pi, & \rightarrow \sin \theta_k = -1, \end{cases} \]  

(100)

where \( p \) is an integer. Introducing phase offsets in the \( k \)’th bit interval, the baseband components from Eq.(95) are expanded to read,

\[ x_i = A_{\epsilon 0} \cos(\theta_k + \varphi_0) \cos a_k \Omega \tau - A_{\epsilon 0} \sin(\theta_k + \varphi_0) \sin a_k \Omega \tau, \]  

(101)

\[ x_q = A_{\epsilon 0} \sin(\theta_k + \varphi_0) \cos a_k \Omega \tau + A_{\epsilon 0} \cos(\theta_k + \varphi_0) \sin a_k \Omega \tau. \]  

(102)

Suppose we are in an even interval. If the \( x_i \) expression above should agree with the similar component in Eq.(81), the time dependency must be contained solely in the \( \sin a_k \Omega \tau \) factor of the second term. Taking the boundary conditions from Eq.(100) into account, the first term will vanish if the phase offset is chosen \( \varphi_0 = \pm \frac{\pi}{2} \). In that case we have,

\[ x_i(kT_b + \tau) \big|_{k \text{ even}} = -A_{\epsilon 0} \sin(\theta_k + \varphi_0) \sin a_k \Omega \tau - A_{\epsilon 0} \cos \theta_k \sin \varphi_0 a_k \sin \Omega \tau. \]  

(103)

Under equal conditions, i.e. without further assumptions, the quadrature component in Eq.(102) reduces to,

\[ x_q(kT_b + \tau) \big|_{k \text{ even}} = A_{\epsilon 0} \sin(\theta_k + \varphi_0) \cos a_k \Omega \tau = A_{\epsilon 0} \cos \theta_k \sin \varphi_0 \cos \Omega \tau. \]  

(104)

Note in particular that the input bit \( a_k \) for the interval in question gets no influence. Furthermore, the result is seen automatically to expose a \( \cos \theta_k \Omega \tau \) time dependency similar to the corresponding quadrature component in Eq.(82). With \( \tau \) constrained to \( \{0,T_b\} \), the last sine and cosine factors in the two equations above are positive, so in comparison to Eqs.(81),(82), the numerical signs are superfluous. The two sets of equation are equal, if the even and odd sign controlling functions from shaped OQPSK are given through

\[ k \text{ even}: \quad b_{\text{evn}}(kT_b + \tau) = b_k = -a_k \cos \theta_k \sin \varphi_0, \quad b_{\text{odd}}(kT_b + \tau) = \cos \theta_k \sin \varphi_0. \]  

(105)

Thus, at an even boundary \( a_k \) translates to the \( b_k \) that controls the sign of the \( b_{\text{evn}}(t) \) function through the following two bit period until the next even boundary. The translation is governed by a \( \cos \theta_k \) factor that updates (integrates) the past input sequence, where \( \theta_k \) is given by

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Eq. (99), and a factor \( \sin \phi_0 \), which represents the initial phase of the signal. At an even boundary the odd function \( b_{\text{odd}}(t) \) does not depend on the actual \( a_k \) value. However, \( b_{\text{odd}}(t) \) must stay constant across an even boundary if the resultant modulated wave should be continuous, and this remains to be proven.

In odd intervals, maintaining the assumption of \( \phi_0 = \pm \frac{1}{2} \pi \), developments similar to the last paragraph now provide,

\[
x_i'(kT_b + \tau) \bigg|_{k \text{ odd}} = A_{x0} \cos (\theta_k + \phi_0) \cos a_k \Omega \tau = -A_{x0} \sin \theta_k \sin \phi_0 \cos \Omega \tau, \quad (106)
\]

\[
x_q'(kT_b + \tau) \bigg|_{k \text{ odd}} = A_{x0} \cos (\theta_k + \phi_0) \sin a_k \Omega \tau = -A_{x0} \sin \theta_k \sin \phi_0 a_k \sin \Omega \tau, \quad (107)
\]

For even intervals, \( b_{\text{even}}(kT_b + \tau) = -\sin \theta_k \sin \phi_0 \), \( b_{\text{odd}}(kT_b + \tau) = b_k = -a_k \sin \theta_k \sin \phi_0 \). (108)

Comparisons between Eqs. (106), (107) and Eqs. (103), (104) show three differences. First, the time dependencies have exchanged form, but this is consistent with the fact that \( |\sin \Omega \tau| \) maps to \( |\cos \Omega \tau| \) or reversely, when the time origin is shifted a quarter period in either direction.

Second, the updating from the past through the offset phase \( \theta_k \) is expressed through \( \sin \theta_k \) instead of \( \cos \theta_k \). This is a consequence of the constraints from Eq. (100).

Third, the updating from the \( a_k \) sequence has moved to the \( b_{\text{odd}}(t) \) function with no impression left on \( b_{\text{even}}(t) \), which here must be proven to stay constant across the bit boundary. To see this, we roll back one bit period using the \( \theta_k \) recursion relation from Eq. (99) and taking the constraints from Eq. (100) into account, i.e.

\[
b_{\text{even}}(t) \bigg|_{k \text{ odd}} = -\sin (\theta_k) \sin \phi_0 = -\sin \left( \theta_{k-1} + a_{k-1} \frac{\pi}{2} \right) \sin \phi_0 = -a_{k-1} \cos \theta_{k-1} \sin \phi_0. \quad (109)
\]

The result agrees with the \( b_{\text{even}}(t) \) value that was inserted in the foregoing even numbered \( k-1 \)th step through the first part of Eq. (105). Similarly, if we roll forward one step, the value inserted in \( b_{\text{odd}}(t) \) transforms,

\[
b_{\text{odd}}(t) \bigg|_{k \text{ odd}} = -a_k \sin (\theta_k) \sin \phi_0 = -a_k \sin \left( \theta_{k+1} - a_k \frac{\pi}{2} \right) \sin \phi_0 = \cos \theta_{k+1} \sin \phi_0, \quad (110)
\]

which corresponds to \( b_{\text{odd}}(t) \) in the last part of Eq. (105). This completes the confirmation of equivalency between MSK and half-sine shaped OQPSK modulations.

Summarizing the results above, the quadrature modulator in Fig. 48 may replace the direct MSK modulator in Fig. 51 and produce the same output, if the MSK bit sequence is translated according to
To exemplify the updating, consider a translation upwards in Fig.47 from $y(t)$. Identifying $\omega_1 \rightarrow a_k=1$ and $\omega_2 \rightarrow a_k=-1$, and using $c_i = -\sin \varphi_0 = 1$ or $c_i = -1$, we get the correct even and odd bit sequences that control the in-phase $x_i$ and quadrature $x_q$ baseband signals by the following table.

Table III Translation of MSK bit sequence to shaped OQPSK in the example of Fig.47.

<table>
<thead>
<tr>
<th>$k$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_k$</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>$\theta_k$</td>
<td>0</td>
<td>$-\pi/2$</td>
<td>$-\pi$</td>
<td>$-\pi/2$</td>
<td>$-\pi$</td>
<td>$-3\pi/2$</td>
<td>$-2\pi$</td>
<td>$-3\pi/2$</td>
<td>$-\pi$</td>
<td>$-\pi/2$</td>
</tr>
<tr>
<td>$b_k = -a_k \cos \theta_k$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$b_k = -a_k \sin \theta_k$</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
I-4 Transmission of Modulated RF-Signals

Bandpass Transmission of Narrowband Signals

Transmission of a modulated real signal $y(t)$ through a linear bandpass filter, which has transfer function $H(\omega)$ and impulse response $h(t)$, follows the rules summarized by Fig. 52a. It is assumed that the filter is physically realizable, so $h(t)$ is real-valued and $H(\omega)$ possesses hermitian symmetry, i.e.

$$h(t) \sim H(\omega) : h(t) \text{ real} \iff H^*(-\omega) = H(\omega).$$

(112)

If the filter and the signal have bandwidths that are small compared to the center and the carrier frequencies, they are called narrowbanded filters and signals respectively. Under such circumstances it may suffice to consider the transmission of a low-frequency envelope signal $\zeta(t)$ through an equivalent low-pass baseband filter $K(\omega)$ as indicated by Fig. 52b. Although the two figures show no conceptual differences, the latter may lead to substantial simplification in the task of working out details in a particular transmission problem.

Fig. 52 (a) Transmission of a modulated signal $y(t)$ through a bandpass filter. (b) Transmission of the complex envelope $\zeta(t)$ to $y(t)$ through the equivalent baseband filter.

The outset for the computational technique to be developed is the phasor representation, where a modulated wave may be expressed as real part of a product including a complex envelope $\zeta(t)$ composed from the real-valued in-phase and quadrature base-band components $x_i(t)$, $x_q(t)$, and a complex carrier $\exp(j\omega_c t)$,

$$y(t) = \Re\{\xi(t)\} = \frac{1}{2}\xi(t) + \frac{1}{2}\xi^*(t)$$

(113)

$$\xi(t) = \zeta(t) e^{j\omega_c t}, \quad \text{where} \quad \zeta(t) = x_i(t) + jx_q(t)$$

(114)

The Fourier transform of the complex envelope $\zeta(t)$ is given through the transforms of the in-phase and quadrature components,

$$\zeta(t) \sim Z(\omega) : Z(\omega) = X_i(\omega) + jX_q(\omega)$$

where $x_i(t) \sim X_i(\omega)$ and $x_q(t) \sim X_q(\omega)$.

(115)

Using the modulating property,
the complex modulated wave gets the transform,
\[ \xi(t) \sim \Xi(\omega) : \Xi(\omega) = Z(\omega - \omega_c) = X_i(\omega - \omega_c) + jX_q(\omega - \omega_c). \]  
(117)

Another basic relationship concerning complex time-functions states,
\[ f(t) \sim F(\omega) \iff f^*(t) \sim F^*(-\omega). \]  
(118)

Applying this result, the Fourier transform of the real-valued modulated wave may be cast in either of the following forms,
\[ y(t) \sim Y(\omega) : \]

\[
Y(\omega) = \frac{1}{2} \Xi(\omega) + \frac{1}{2} \Xi^*(-\omega) \quad (a) \\
= \frac{1}{2} Z(\omega - \omega_c) + \frac{1}{2} Z^*(-\omega - \omega_c) \quad (b) \\
= \frac{1}{2} X_i(\omega - \omega_c) + \frac{j}{2} X_q(\omega - \omega_c) + \frac{1}{2} X_i^*(-\omega - \omega_c) - \frac{j}{2} X_q^*(-\omega - \omega_c) \quad (c) \\
= \frac{1}{2} \left[ X_i(\omega - \omega_c) + jX_q(\omega - \omega_c) \right] + \frac{1}{2} \left[ X_i(\omega + \omega_c) - jX_q(\omega + \omega_c) \right]. \quad (d)
\]

The last expression follows from the fact that \(x_i(t)\) and \(x_q(t)\) are real-valued functions, i.e.

Fig.53 Definition of the equivalent baseband transfer function \(K(\hat{\omega})\) to the bandpass transfer function \(H(\omega)\). Note the phase function adjustment to zero phase at \(\hat{\omega}=0\).
A bandpass filter characteristic where Eq.(112) applies is sketched in Fig.53(a). The equivalent baseband filter is taken from the positive frequency part of the bandpass characteristic as indicated by Fig.53(b), that is

\[ K(\hat{\omega}) = \begin{cases} 
H(\hat{\omega} + \omega_c) e^{-j\theta_c}, & \hat{\omega} \geq -\omega_c \\
0, & \hat{\omega} < -\omega_c 
\end{cases} \quad \text{or} \quad K(\omega - \omega_c) = \begin{cases} 
H(\omega) e^{-j\theta_c}, & \omega \geq 0 \\
0, & \omega < 0 
\end{cases} \]  

(121)

The first version emphasizes the baseband nature of K through the baseband angular frequency variable \( \hat{\omega} = \omega - \omega_c \). The exponential factor adjusts the phase of the baseband filter to become zero at \( \hat{\omega} = 0 \). In terms of the baseband equivalent, the bandpass transfer function is expressed,

\[ H(\omega) = K(\omega - \omega_c) e^{j\theta_c} + K^*(-\omega - \omega_c) e^{-j\theta_c}. \]  

(122)

and by Eq.(119)b, the bandpass filter output gets the Fourier transform,

\[ W(\omega) = H(\omega) Y(\omega) \]

\[ = \frac{1}{2} \left\{ K(\omega - \omega_c) e^{j\theta_c} + K^*(-\omega - \omega_c) e^{-j\theta_c} \right\} \left\{ Z(\omega - \omega_c) + Z^*(-\omega - \omega_c) \right\} \]

\[ = \frac{1}{2} K(\omega - \omega_c) Z(\omega - \omega_c) e^{j\theta_c} + \frac{1}{2} K^*(-\omega - \omega_c) Z^*(-\omega - \omega_c) e^{-j\theta_c} \]

\[ + \frac{1}{2} K^*(-\omega - \omega_c) Z^*(-\omega - \omega_c) e^{j\theta_c} + \frac{1}{2} K^*(-\omega - \omega_c) Z(\omega - \omega_c) e^{-j\theta_c}. \]  

(123)

Fig.54 One of the two product terms, here the absolute value of \( K(\omega - \omega_c)Z^*(-\omega - \omega_c) \), which must be ignorable in the narrowband approximation based on Eq.(123).

The condition we are seeking allows us to disregard the two last terms in Eq.(123). This restricts the signal spectrum as demonstrated by Fig.54. To neglect the last terms, the original signal must be narrowbanded, i.e. having a bandwidth less than the carrier frequency. In that case Eq.(123) is approximated.

\[ J.Vidkjær \]
\[ W(\omega) = \frac{1}{2} K(\omega - \omega_c) Z(\omega - \omega_c) e^{j\theta_c} + \frac{1}{2} K^*(\omega - \omega_c) Z^*(\omega - \omega_c) e^{-j\theta_c} \quad (a) \]

\[ - \frac{1}{2} \Psi(\omega - \omega_c) e^{j\theta_c} + \frac{1}{2} \Psi^*(\omega - \omega_c) e^{-j\theta_c} \quad (b) \]

where the function \( \Psi(\omega) \) is defined by,

\[ \Psi(\omega) = K(\omega) Z(\omega) \quad (125) \]

Using (116), the modulated wave may now be written,

\[ w(t) \rightarrow W(\omega) : \quad w(t) = Re \left\{ \psi(t) e^{j\theta_c} e^{j\omega_c t} \right\} \quad (126) \]

Here the two exponentials hold the carrier and the fixed phaseshift \( \theta_c \) - known as the phase delay - that applies to the transmission through the narrowband filter at the carrier frequency. The complex envelope of the output signal is given through Eq.(125), which in time domain corresponds to

\[ \psi(t) \rightarrow \Psi(\omega) : \quad \psi(t) = \kappa(t) \ast \zeta(t) . \quad (127) \]

\( \kappa(t) \) is the impulse response of the equivalent baseband filter, so the output envelope becomes the input envelope transmitted through the equivalent baseband filter. Similar to the input modulated wave, \( y(t) \), the filter output \( w(t) \) gets the quadrature representation

\[ w(t) = w_q(t) \cos(\omega_c t + \theta_c + \varphi_0) - w_q(t) \sin(\omega_c t + \theta_c + \varphi_0) \quad (128) \]

where \( w_q(t) = Re\{\psi(t)\} , \quad w_q(t) = Im\{\psi(t)\} . \)

Once the complex envelope is found, the traditional envelope representation may be calculated from the expressions,

\[ w(t) = A_o(t) \cos(\omega_c t + \theta(t) + \theta_c + \varphi_0) , \quad \text{with} \]

\[ A_o(t) = \sqrt{w_q^2(t) + w_q^2(t)} \quad \theta(t) = \tan^{-1} \left( \frac{w_q(t)}{w_q(t)} \right) \quad (129) \]

Care must be exercised when the results of the developments above are used in practice. Due to the definition in Eq.(121), the equivalent baseband filter does not automatically possess the hermitian symmetry from Eq.(112) so customary \( K^*(-\omega) = K(\omega) \) should not be expected. Thus, the baseband equivalent to a physically realizable bandpass filter is not necessarily realizable too. If hermitian symmetry does not apply to the equivalent baseband transfer function, it gets a complex impulse response and real valued signals are transferred to complex signals. Under such circumstances the equivalent baseband considerations may no longer serve as a simplifying tool for getting insight and making simple analysis, albeit the technique is still valid and may be used in computer simulations or other signal processing.

\( \text{J.Vidkjær} \)
tasks. Below we exemplify two types of problems that have or may approximate the required symmetry, so the baseband transfer functions behave like conventional linear circuits.

Example I-4-1 (ideal bandpass filter transmission)

The simplified bandpass filter in Fig.55 has a transfer function magnitude which is approximated by a constant and the phase is taken to be linear within the passband.

Fig.55  Simplified bandpass filter where the transfer function magnitude may be approximated by a constant and the phase is taken to be linear within the passband.

The simplified bandpass filter in Fig.55 has a transfer function magnitude which is taken constant $H_c$ throughout the passband. Its phase is supposed to be so linear that it suffices to substitute it by the constant and 1st order terms in a Taylor series, i.e.

$$H_{NB}(\omega) = \begin{cases} H_c \ e^{j \theta(\omega)}, & |\omega + \omega_c| < \frac{1}{2} W, \\ 0 & \text{otherwise}, \end{cases} \quad \theta(\omega) = \pm \theta_c - \tau_{dly}(\omega_c) \ [\omega + \omega_c],$$

(130)

The negative value of the phase derivative with respect to angular frequency, $\tau_{dly}$, is known as the filter group delay, which is conveyed to the equivalent baseband filter. If an input signal is bandlimited to stay within the passband of the filter, the complex envelope $Z_{BL}(\omega)$ is transferred through the filter according to Eq.(125),

$$K(\omega) \approx H_c \ e^{-j \tau_{dly}(\omega)} \omega \quad \Rightarrow \quad \Psi(\omega) = H_c \ Z_{BL}(\omega) \ e^{-j \tau_{dly}(\omega)}$$

(131)

From the time shifting property of the Fourier transform,

$$f(t) \ast F(\omega) \quad \Rightarrow \quad \mathcal{F}^{-1} \left\{ F(\omega) \ e^{j \omega t_o} \right\} = f(t + t_o),$$

(132)

it is seen, that the baseband filter delays the complex envelope by its group delay. The output from the filter is given through,

$$w(t) \ast W(\omega) : \quad w(t) = H_c \ \text{Re} \left\{ \zeta(t - \tau_{dly}) \ e^{j \theta_c} \ e^{j \omega t} \right\}$$

(133)
In this case the filter transfers a real-valued signal in ordinary envelope form by

\[ y(t) = A(t) \cos(\omega_c t + \varphi(t) + \varphi_o) \]

\[ \Rightarrow w(t) = H_c A(t - \tau_{dl}) \cos(\omega_c t + \theta_c + \varphi(t - \tau_{dl}) + \varphi_o) \]  

(134)

Example I-4-1 end

Example I-4-2 (tuned circuit transmission)

[Diagram of equivalent circuit for a single-tuned rudimentary amplifier and the corresponding baseband low-pass circuit. The figures show the carrier pulse response and the equivalent low-pass envelope response.]

Hermitian symmetry in the equivalent baseband transfer function may also be present if the filter is symmetric and apply to the type of narrowband approximation that is used in circuit theory. A simple example is here the transmission of a carrier pulse through the

Fig.56 Equivalent circuit for a single-tuned rudimentary amplifier and the corresponding baseband low-pass circuit. The figures show the carrier pulse response and the equivalent low-pass envelope response.

12) While the frequency transformation between baseband and passband in signal processing is linear due to Eq.(116), the transformations between low-pass and bandpass in circuit theory is nonlinear,

\[ \frac{\Delta \omega}{W_{LP}} \approx \frac{\omega_o}{W_{BP}} \left( \frac{\omega - \omega_o}{\omega_o} \right) \approx \frac{2 \Delta \omega}{W_{BP}} \left( \frac{\omega - \omega_o}{W_{BP}} \right). \]

\( W_{LP} \) and \( W_{BP} \) are the 3dB bandwidths in low-pass and bandpass respectively. The last, linear approximation is a 1st order Taylor expansion around \( \omega_o \). If it applies beyond the bandwidth \( W_{BP} \), the circuit is said to be narrowbanded. This property is discussed further in Chapter II.

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The center frequency of the parallel circuit is supposed to equal the carrier frequency and the amplifier has the bandpass transfer function,

\[
H(\omega) = \frac{V_{out}}{V_{in}} = -g_m \frac{R}{1 + jQ\left(\frac{\omega - \omega_0}{\omega_0}\right)} \approx \frac{-g_m R}{1 + jRC(\omega - \omega_0)},
\]

where \( \omega_0 = \frac{1}{\sqrt{LC}} \), \( Q = \frac{\omega_0}{W_{BP}} = RC\omega_0 \).

The last expression is the narrowband approximation for the circuit and gives the transfer function the desired symmetry around the center frequency \( \omega_0 \). If \( Q \gg 1 \), the approximation is useful across the bandwidth of the amplifier. The baseband equivalent circuit becomes a simple low-pass amplifier with 3dB bandwidth that is half the bandwidth of the tuned circuit,

\[
K(\omega) = \frac{-g_m R}{1 + j2RC\omega} = \frac{-g_m R}{1 + j\omega/W_{LP}}, \quad \text{where} \quad W_{LP} = \frac{1}{2RC} = \frac{1}{2} W_{BP}.
\]

The pulse response of the equivalent low-pass amplifier is shown in the right half of Fig.56. If \( p(t) \) denotes the input pulse in Fig.56e, an example of a complex envelopes in a OQPSK modulated input signal to the amplifier could be,

\[
\zeta(t)|_{OQPSK} = e^{j\frac{\pi}{4}} \sum_{k=0}^{3} j^k p(t - kT_b)
\]

Calling the pulse response from Fig.56f \( r(t) \), the complex and ordinary envelopes in response to the OQPSK signal are

\[
\Psi(t)|_{OQPSK} = e^{j\frac{\pi}{4}} \sum_{k=0}^{3} j^k r(t - kT_b), \quad A_o(t)|_{OQPSK} = \sqrt{\sum_{k=0}^{3} r^2(t - kT_b)}.
\]

Fig.57 Response of the rudimentary bandpass amplifier to a OQPSK modulated signal. The upper curve shows the envelope calculated by Eq.(138). The lower curve shows a simulated response.
The ordinary envelope is shown in Fig. 57 and compared to the pertinent bandpass waveform from a simulation.

Example I-4-2 end

The two examples above are simple cases where the equivalent baseband technique could be conducted analytically. In a wider perspective the method gives the foundation for studying basic modulation and coding properties of many RF-communication system while staying in baseband. As sketched in Fig. 58, all that is needed for the purpose is the equivalent baseband filter. In computer simulation the filter no longer needs to be simple. A drawback of the approach is, however, that it is hard or impossible to include nonlinear effects. They will often set the practical limits on performance in a RF-system.

Fig. 58 Block-diagram representation of a communication system where the RF-part is condensed in an equivalent baseband filter.
I-5 Receiver and Transmitter Structures

While the modulation methods and standards that are used in practice have undergone several significant changes during the history of radio communication, the basic RF architectures in transmitters and receivers have stayed remarkably constant over the same period of time. Below we shall present and discuss the most important aspects of common transmitter and receiver designs.

Heterodyning

![Diagram of heterodyning](image)

Fig.59 Heterodyning examples. Frequency is stepped up in the transmitter (a). Incoming signals are transferred to the intermediate frequency in the receiver (b), where the IF filter separates the channels.

Information conveyed by a modulated signal is kept in the envelope of the carrier, in the deviation of phase from the carrier phase, or in both. The carrier frequency moves the modulated signal to the proper frequency range for transmission, but its actual value contributes nothing to the message being send or received. To ease processing and filtering the carrier frequency may undergo several substitutions inside transmitters and receivers from it leaves the modulator to it reaches the demodulator. In case of coherent demodulation, it is the carrier that applies to the demodulator input that must be recovered. The principle of changing carrier frequencies is called heterodyning and the technique is to multiply the modulated signal by another sinusoidal carrier of fixed amplitude and frequency. Using the frequency $\omega_{LO}$ - LO stands for local oscillator - we get in time domain with an arbitrarily modulated signal $y(t)$,
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As seen, the multiplication creates two signals modulated by the original baseband components \( x_i(t) \) and \( x_q(t) \). One of the signals must be selected by subsequent bandpass filtering either around the resultant sum or difference between the original and the local oscillator frequencies. Bandpass filtering must always accompany the multiplication, i.e. mixing, as seen in the examples of Fig.59. Using sums the process is called up-conversion as the resultant carrier frequency \( \omega_c + \omega_{LO} \) is higher than the original carrier frequency \( \omega_c \). By down-conversion the resultant carrier becomes lower than the original one. To get this, the difference frequency term must be chosen, but it depends on the local oscillator frequency whether we get down or up conversion to a resultant frequency of \( |\omega_c - \omega_{LO}| \).

No consequences may be ascribed to the sign of the new carrier frequency \( \omega_c - \omega_{LO} \) as it follows from Eq.(139). However, the two choices of \( \omega_c, \omega_{LO} \) that lead to the same absolute difference has practical implications in receivers. They are most clearly revealed in the frequency domain. To see this, a signal with in-phase modulation alone like AM is considered. Other modulations follow similar lines, but employ the full notation of Eq.(119), which in the present situation adds nothing to enlighten the problem. Eq.(116) implies

\[
\hat{y}(t) = y(t) \cos \omega_{LO} t = \left[ x_i(t) \cos \omega_c t - x_q(t) \sin \omega_c t \right] \cos \omega_{LO} t \tag{a}
\]

\[
= x_i(t) \cos \omega_c t \cos \omega_{LO} t - x_q(t) \sin \omega_c t \cos \omega_{LO} t \tag{b}
\]

\[
= \frac{1}{2} x_i(t) \cos(\omega_c - \omega_{LO}) t - \frac{1}{2} x_q(t) \sin(\omega_c - \omega_{LO}) t \tag{c}
\]

\[
+ \frac{1}{2} x_i(t) \cos(\omega_c + \omega_{LO}) t - \frac{1}{2} x_q(t) \sin(\omega_c + \omega_{LO}) t \tag{d}
\]

Fig.60. shows the corresponding spectra in case of \( \omega_c < \omega_{LO} \) and here the terms in Eq.(141) are written in frequency order.

Thereby the translation in frequency becomes, cf. Eq.(119),

\[
Y(\omega) = \frac{1}{2} X_i(\omega - \omega_c) + X_q(\omega + \omega_c) \quad \Rightarrow
\]

\[
\hat{Y}(\omega) = \frac{1}{2} Y(\omega + \omega_{LO}) + \frac{1}{2} Y(\omega - \omega_{LO}) \quad \Rightarrow \tag{141}
\]

\[
= \frac{1}{4} X_i(\omega + \omega_c + \omega_{LO}) + \frac{1}{4} X_i(\omega + \omega_c - \omega_{LO}) + \frac{1}{4} X_q(\omega - \omega_c + \omega_{LO}) + \frac{1}{4} X_q(\omega - \omega_c - \omega_{LO})
\]

Fig.60. shows the corresponding spectra in case of \( \omega_c < \omega_{LO} \) and here the terms in Eq.(141) are written in frequency order.

Receivers must select among tightly spaced channels by filtering with filter bandwidths close to the signal bandwidth and simultaneously good suppression of neighboring channels. These requirements are unrealistic for RF filters, if they also should be tunable in frequency. To select a given channel a filter of fixed frequency - the intermediate frequency, IF, - is used instead. By heterodyning, all the incoming signals are moved in frequency until the desired one coincide with the fixed intermediate frequency. Unfortunately, there are two
I-5 Receiver and Transmitter Structures

Fig. 60 Frequency translation by heterodyning with $\omega_c - \omega_{LO}$. The arrows show how the terms appear in the translation by Eq. (141).

Input signals that may pass the IF filter, one at the carrier frequency that translates to the requested IF, taken positive or negative, and one translating to the IF with opposite sign. The latter is called the image response and the corresponding frequency is denoted $\omega_{ci}$. In case the intermediate frequency is a difference frequency we have,

$$\omega_{IF} = |\omega_{c,ci} - \omega_{LO}| : \begin{cases} \omega_c = \omega_{LO} - \omega_{IF} \\ \omega_{ci} = \omega_{LO} + \omega_{IF} \end{cases} \text{ or } \begin{cases} \omega_c = \omega_{LO} + \omega_{IF} \\ \omega_{ci} = \omega_{LO} - \omega_{IF} \end{cases} \quad (142)$$

Fig. 61 Image response creation. The upper diagram shows the desired IF translation. The lower diagram shows the image from a signal $2\omega_{IF}$ apart from the desired signal.
Fig. 61 shows an example corresponding to Eq. (142), where the upper part is the desired translation to the intermediate frequency while the lower part demonstrates the corresponding image response. Had we chosen the latter as the signal being aimed upon, the upper signal would represent the image. In both cases, the distance between the desired signal and its image is twice the intermediate frequency. Down-conversion as above is a usual choice, but in case the sum frequency is chosen as IF, image responses occur at frequencies given by

\[
\omega_{IF} = \begin{cases} 
\omega_c + \omega_{LO} \\
\omega_c - \omega_{LO} 
\end{cases}
\quad \text{or} \quad \begin{cases} 
\omega_c = \omega_{IF} - \omega_{LO} \\
\omega_c = \omega_{IF} - \omega_{LO} \\
\omega_c = \omega_{IF} - \omega_{LO} - \omega_{IF} 
\end{cases}
\] (143)

There are no simpler mean to avoid the image response in receivers than suppressing the image signal before it reaches the IF mixer. This is the role of the RF filter in front of the receiver that is shown in Fig. 59. The RF filter is a bandpass filter having passband around the required signal. Its operation is illuminated by Fig. 62 under conditions equal to the lower part of Fig. 61. As indicated, the RF filter is commonly less selective than the IF filter. If the receiver covers a broad input frequency range, the RF filter is tunable and tracks the local oscillator in a distance of $\omega_{IF}$. Tunable filter must be simple at the expense of selectivity. Receivers of the type in Fig. 59 are called super heterodyne receivers. The principle was patented by E.H. Armstrong in 1917, and - without any doubt - this is still the most common radio receiver structure.

The smaller a desired passband is compared to the center frequency, the more difficult is it to build a selective filter. This fact influences the characteristics of both the RF and the IF filters, and it is the reason why the intermediate frequency commonly is chosen smaller than the RF input, since it is the sharpness of the IF filter that determines how good different channel are separated. However, it is more easy to get good image response rejection using the IF as high as possible, so a compromise between the two concerns must be made.
Alternatively, there may be two IF stages as sketched in Fig.63, where the first stage gives image response suppression and the second one makes channel separation. This is called a double conversion receiver.

**Image Response Eliminations**

Alternatives to the image response suppression techniques above are solutions that try to overcome the problem by other circuit structures. One method is to employ a so-called image rejection mixer, which has a structure like Fig.64. We have already seen the essential part of the scheme before, namely as the SSB demodulator circuit in Fig.17. Recall, that in the SSB demodulation case, the role of the circuit was to decide whether to detect signals from the upper or lower sideband frequency range, i.e. frequency bands placed symmetrically around the carrier. This is equivalent to the problem where we have to decide between the desired response and its image, which are placed symmetrically around the local oscillator LO frequency at distances of IF. The sign of the $90^\circ$ phaseshifter is now to decide whether the input RF signal is $\omega_{c-}\omega_{LO}$ or $\omega_{c+}\omega_{LO}$. Compared to the SSB demodulator case, it is not directly required to include lowpass filters, since the IF filter at the output of the mixer is supposed to reject frequency components around twice the LO frequency. Note, however, that there could be other reasons for maintaining the lowpass filters as well as the RF filter in a practical amplifier.

13 Noise limits and suppression of spurious frequency components from nonlinearities require often additional filtering. Both type of problems are dealt with later.
A more radical approach to the image response problem is to convert directly from RF to baseband without any intermediate frequency. This is the structure in the basic modulator/demodulator pair that was introduced by Fig.13, but at that time no attention was given to the problem of choosing only one channel among more. A receiver following the scheme, as sketched in Fig.65, is called direct conversion or homodyne receiver. Compared to the heterodyne case, direct conversion has no IF, so the RF signal is its own image, and it will not disturb the reception. Instead of separating different channels by a bandpass IF filter, channels separation is now made by lowpass filtering with bandwidth equal to the signal bandwidth. In comparison with the basic demodulator scheme this is a strengthening of the filter characteristics since, originally, the LP filters should only suppress components at second harmonics in the carrier frequency. Nevertheless, the LP filtering is considered as another advantage of the direct conversion principle, since LP filters with the required characteristics may be suited for digital implementations in integrated circuits.

In spite of the obvious advantages of the direct conversion principle, which has been known for as long time as the heterodyne principle, it is still not in widespread use. There are several reasons for that, most of them related to the fact that the local oscillator frequency must equal the carrier frequency of the RF input signal. In many practical radio systems, the latter may be more than 90dB less than the oscillator level and the requirements for isolating the local oscillator from the RF signal path prior to mixing must be even better. Otherwise, the LO signal may be radiated from the antenna and disturb other receivers in the same communication system. Alternatively, the leaking oscillator signal may mix with itself to produce DC terms in the mixer outputs that can overwhelm baseband signal DC terms or debias subsequent circuits, so the demodulation becomes erroneous. For communication systems with limited performance requirements, however, direct conversion provides a simple way of building receivers. Example I-5-1 describes one of the first direct conversion receiver structures with internal channel separation filters that was commercially available as an IC. Presently many efforts are given to improve RF-IC processes and design method to meet the top requirements in mobile communications using direct conversion receivers [8].

Fig.65 Direct conversion receiver structure. The LP filters have bandwidths equal to the signal bandwidth in each channel. Local oscillator leakage paths are harmful to performance.
Example I-5-1 (direct conversion FSK receiver IC)

The FSK receiver circuit above was introduced for pager applications where the requirements with respect to frequency, signalling speed, and input power sensitivity are relatively small compared to mobile phone standards. To see the basic principle of the circuit, assume that incoming logical ones and zeros are RF signals of frequencies \( \Delta \omega \) above or below the carrier frequency \( \omega_c \), i.e.

\[
\begin{align*}
    s_1 & = \cos(\omega_c + \Delta \omega), \\
    s_0 & = \cos(\omega_c - \Delta \omega).
\end{align*}
\]  

\text{(144)}

The local oscillator is tuned to the carrier, but its phase \( \phi \) may be arbitrary compared to the input signal. In the upper, inphase branch of the receiver, mixing with the local oscillator provides

\[
I_{\text{tot}} = \cos(\omega_c + \Delta \omega) \cos(\omega_c + \phi) - \frac{1}{2} \cos(\pm \Delta \omega - \phi) + \frac{1}{2} \cos(2\omega_c + \ldots)
\]  

\text{(145)}

After lowpass filtering, taking into account that

\[
\cos a = \cos(-a),
\]  

\text{(146)}

the inphase channel presents the signal,

\[
I_{\text{LP}} = \frac{1}{2} \cos(\Delta \omega + \phi),
\]  

\text{(147)}

to a limiter circuit, that converts the input sinusoidal to a square wave with the same phase. Observe that the phase of the local oscillator is contained in the I signal phase with sign determined by the incoming bit signal. In the quadrature branch we get,

\( J.Vidkjær \)
\[ Q_{\text{tot}} = -\cos(\omega_c + \Delta \omega) \sin(\omega_c + \varphi) = \frac{1}{2} \sin(\Delta \omega - \varphi) - \sin(2 \omega_s + ...) \] (148)

After lowpass filtering, the quadrature channel signal gets sign corresponding to incoming signal according to

\[ Q_{LP} = \frac{1}{2} \sin(\Delta \omega + \varphi) . \] (149)

We notice that the phase of the local oscillator is contained in the quadrature signal the same way as it was in the inphase signal. The decision of whether a logical one or a logical zero was received is therefore a question on whether or not the Q signal leads or lags the I signal. In its simplest form, this may be done by a D flip-flop as shown in the figure.

Example I-5-1 end
Problems

P.I-1

An AM-transmitter has an unmodulated carrier of 50 kW. It is modulated by a sinusoidal signal of maximum baseband amplitude, which creates upper and lower sideband components that each are 40% of the carrier amplitude. What is the modulation index of the baseband signal and what is the total output power?

P.I-2

Fig.67 shows the principle of a FM modulator, where details of the narrowband modulator correspond to Fig.14 and Fig.15. It is called an Armstrong modulator after the inventor. The baseband signal $x(t)$ has frequency components from 50Hz to 15kHz. The FM output must have a peak-frequency deviation of $\Delta f_{\text{max}}=75 \text{ kHz}$ around a 96 MHz carrier.

The first block in the transmitter is a narrowband FM modulator where the baseband signal modulates a 200 kHz carrier with a maximum modulation index of $\beta=0.5$. The output carrier and peak-frequency deviation are adjusted to the final requirements using frequency multipliers $M_1, M_2$ and mixing with a sinusoid of frequency $f_o[\text{Hz}]$. Find a combination of $M_1$ and $f_o$ that produce the required output using $M_2=48$ and indicate the frequency range of the bandpass filter.

P.I-3

The quadrature demodulator in Fig.68 may reconstruct the in-phase and quadrature components $x_i(t), x_q(t)$. There will be a cross-over between the two - expressed by factor $\varepsilon$ - if the local oscillator is not exactly synchronized. What is the greatest phase synchronization error $\theta_s$, if the cross-over should be less than -40dB?
Suppose the synchronization is perfect with respect to the in-phase component, but the demodulator has a quadrature error $\theta_q$. How big may this be if the resultant cross-over should still lie below -40dB?

**P.I-4**

A FSK modulated signal is transmitted at the rate 19000 bps (bit per second). Logical one corresponds to $f_1 = 2.0095$ Mhz and logical zero to $f_2 = 1.9905$ Mhz. The signal is received at a signal-to-noise ratio of $E_b/\eta \sim 8$ dB. What is the bit error rate, BER, in synchronous demodulation?

The signal frequencies are changed to $f_1 = 2.0063$ Mhz and $f_2 = 1.9937$ Mhz respectively. What is the new bit error rate?

**P.I-5**

A binary FSK modulated signal has the bit energy $E_b$, a bit rate fairly below the carrier, $1/T_b < f_c$, and the modulation index $h=1$ (Sunde’s FSK)

Find expressions for the in-phase and quadrature components in the signal and suggest a block scheme for a modulator. Calculate the power spectrum of the signal if logical 1’s and 0’s are equally probable.

**P.I-6**

[Diagram of a QPSK modulated signal received by a simple single-tuned amplifier]

A QPSK modulated signal is received by the simple single-tuned amplifier in Fig.69. The signal has the symbol period $T_s=31.52$ ns and the amplifier is tuned to the carrier frequency $f_c=476$ Mhz. At the input terminal the signal is $v_{in} = 30 \mu V_{RMS}$, the transistor has $g_m = 20$ mS, and the load is $R_L = 120 \Omega$ with a Q-factor of 10.

Draw the real envelope to the output signal $v_{out}(t)$ for an input signal corresponding to the bit sequence 110011011000.
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